Problems I

1) Consider an electron storage ring with a **circumference** of **850m** and **energy** of **6GeV**. If the maximum **bending field** available is **0.854T**, what is the percentage of the circumference occupied by dipoles? With dipoles of **2.3m** long, find the integrated dipole strength, the bending angle and the number of dipoles.

For 6 GeV electrons $\beta \simeq 1$

$$B\rho \; (\mathrm{T.m}) = \frac{10}{2.998} \beta E^{(\mathrm{GeV})}$$

$$\rho = 23.435 \; \mathrm{m}$$

The percentage occupied by dipoles is

$$\frac{2\pi\rho}{C} = 17.3 \; \%$$

The integ. field $B\ell = 1.9648 \; \mathrm{Tm}$

The bend. angle $\theta = \frac{\ell}{\rho} = 0.098 \; \mathrm{rad} = 5.623°$

$$N = \frac{2\pi}{\theta} \simeq 64 \text{ dipoles}$$

Problems I

2) Trace the poles of a decapole and dodecapole magnet. What is the angle between the center of each pole in each case? Derive the angle between the poles of general **2n**-pole magnets?

- The decapole (10 poles) with an angle $\mathbf{2\pi/10 = \pi/5 = 36^o}$ between the poles
- The dodecapole (12 poles) with an angle $\mathbf{2\pi/12 = \pi/6 = 30^o}$ between the poles
- The angle of a **2n**-pole will be $\mathbf{2\pi/(2n) = \pi/n}$

Problems I

3) Use the expansion of the scalar potential in polar coordinates in order to show that the potential is symmetric by a rotation of $\boldsymbol{\pi}$. Prove that the first allowed multi-pole for a normal quadrupole magnet is a 12-pole ($\mathbf{b_6}$), the second a 20-pole ($\mathbf{b_{10}}$), etc. Is there a general rule for all multi-pole magnets?

$$\Phi(r, \theta) \propto e^{in\theta}$$

$$\Phi(r, \theta+\pi) \propto e^{in(\theta+\pi)} = e^{in\theta} e^{in\pi} = (-1)^n e^{in\theta}$$

Thus the potential is symmetric for even and anti-symmetric for odd multipoles

For a quad $\quad \Phi(0) = -\Phi(\pi/2) = \Phi(\pi) = -\Phi(3\pi/2)$

$$\cos(n\pi/2) = -1 \rightarrow n\pi/2 = (2k+1)\pi \rightarrow$$

$$n = 2(2k+1) \qquad \begin{array}{l} k=0 \rightarrow n=2 \\ k=1 \rightarrow n=6 \\ k=2 \rightarrow n=10 \end{array}$$

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By rotating a 2m-pole ~~the~~ by an angle $\frac{\pi}{m}$, the field should be inversed:

$$\cos(n\,\pi/m) = -1 \rightsquigarrow$$

$$n\pi/m = (2k+1)\,\pi \rightsquigarrow$$

$$\boxed{n = (2k+1)\,m}$$

4) Prove that the transfer matrices of two symmetric cells and of one cell with mirror symmetry have their determinant equal to 1. Derive the transfer matrix of a particle moving in the opposite direction in the two above cases.

For a symmetric cell

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(M) = 1.$$

So

$$\det(M^2) = [\det(M)]^2 = 1.$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{pmatrix}$$

$$\mathrm{Det}(M) = (a^2+bc)(cb+d^2) - (ab+bd)(ac+cd)$$

$$= \cancel{a^2cb} + b^2c^2 + a^2d^2 + \cancel{bcd^2} - \cancel{a^2bc} - abcd - abcd - \cancel{bcd^2}$$

$$= (ad-bc)^2 = \det(M) = 1\,!$$

4) Prove that the transfer matrices of two symmetric cells and of one cell with mirror symmetry have their determinant equal to 1. Derive the transfer matrix of a particle moving in the opposite direction in the two above cases.

For a mirror symmetric cell

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \text{ with } \det(M_r) = ad - bc = \det(M) = 1.$$

$$\det(M_r \cdot M) = \det(M_r) \cdot \det(M) = 1$$

OR

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} ad+bc & 2bd \\ 2ac & ad+bc \end{pmatrix} =$$

$$= (ad+bc)^2 - 4abcd = (ad-bc)^2 = \det(M) = 1$$

For a particle moving in the opposite direction

$$M^i_{tot} = (M_{tot})^{-1} = \begin{cases} \text{symmetric} & \begin{pmatrix} cb+d^2 & -b(a+d) \\ -c(a+d) & bc+a^2 \end{pmatrix} \\ \text{mirror symmetric} & \begin{pmatrix} ad+bc & -2bd \\ -2ac & ad+bc \end{pmatrix} \end{cases}$$

Problems I

5) Find the focal length of a thin focusing and defocusing quadrupole. To do so, consider an incoming parallel beam (in x or in y depending on the quad) and propagate it using the quad and a drift, and find the drift length in order to get **0** displacement. Do the same for both planes for a doublet formed by the two quads, with distance **L** between them.

$$\begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \mp \frac{L}{f} & L \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Thus $0 = \left(1 \mp \frac{L}{f}\right) x \rightarrow$

$$1 \mp \frac{L}{f} = 0 \rightarrow \boxed{f = \pm L}$$

Problems I

5) Find the focal length of a thin focusing and defocusing quadrupole. To do so, consider an incoming parallel beam (in x or in y depending on the quad) and propagate it using the quad and a drift, and find the drift length in order to get **0** displacement. Do the same for both planes for a doublet formed by the two quads, with distance **L** between them.

For a doublet

$$\begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f^*} & 1-\frac{L}{f_2} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{d}{f^*}-\frac{L}{f_1} & L+d-\frac{dL}{f_2} \\ -\frac{1}{f^*} & 1-\frac{L}{f_2} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Thus,

$$0 = \left(1-\frac{d}{f^*}-\frac{L}{f_1}\right) x$$

$$1-\frac{d}{f^*}-\frac{L}{f_1} = 0 \rightsquigarrow \frac{1}{d}\left(1-\frac{L}{f_1}\right) = \frac{1}{f^*}$$

setting $f_1 = -f_2 = f$

$$\boxed{\frac{1}{f^*} = \frac{1}{d}\left(1-\frac{L}{f}\right)}$$

Problems II

6) Write the transfer matrix of a FODO cell for which the integrated quad strength is **1/f = ± 1/2f** and the drift has distance **l**. For this quad and considering the propagation of optics functions in a symmetric cell obtain an expression for the phase advance **μ** and the beta function **β**, at the focusing quad. Without matrix multiplication do the same for the defocusing quad. For numerical evaluation you will need that **Bρ = 26.68 Tm**, the quad length **0.509 m** the quadrupole gradient **12 T/m** and the distance between quads **6.545 m**.

For the FODO cell with $\pm 1/f \rightarrow \pm 1/2f$

$$M_{FODO} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & 2L\left(1 + \frac{L}{4f}\right) \\ \frac{L}{8f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} \end{pmatrix}$$

For a symmetric cell

$$Tr(M) = 2\cos\mu \rightarrow 2\cos\mu = \cancel{2} - \frac{L^2}{4f^2}$$

$$\mu = \arccos\left[1 - \frac{L^2}{8f^2}\right]$$

$$f = \frac{1}{k\ell} = \frac{B\rho}{G\,\ell} = 4.368\,m$$

$$\mu \simeq 44°$$

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$$\beta \sin\mu = 2L\left(1+\frac{L}{4f}\right) \rightarrow$$

$$\beta = \frac{2L}{\sin\mu}\left(1+\frac{L}{4f}\right) \simeq 25.3\,\mathrm{m}$$

For the phase advance in the def. quad we just have to consider half of the cell

$$\mu' = \mu/2 \simeq 22^\circ$$

For the beta function, we just have to reverse the sign of f:

$$\beta^{def} = \frac{2L}{\sin\mu}\left(1-\frac{L}{4f}\right) \simeq 11.38\,\mathrm{m}$$

Problems II

7) Write the **(α,β,γ,D,D')** functions propagation along a drift. Modify accordingly the formulas, assuming a symmetry point at the entrance of the drift. What is the beta function at the entrance of the drift in order to have a minimum value at the exit? For the numerical evaluation assume a **3m-**long drift.

$$\beta(s) = \beta_0 - 2s\alpha_0 + s^2\gamma_0$$
$$\alpha(s) = \alpha_0 - s\gamma_0$$
$$\gamma(s) = \gamma_0$$

Starting at a mirror symmetric point: $\alpha_0 = 0$

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$
$$\alpha(s) = \frac{-s}{\beta_0}$$
$$\gamma(s) = \frac{1}{\beta_0}$$

$$\beta_{exit} = \beta_0 + \frac{d^2}{\beta_0}$$

$$\frac{d\beta_{exit}}{d\beta_0} = 1 - \frac{d^2}{\beta_0^2} = 0 \quad \text{(minimum)}$$

Thus $\boxed{\beta_0 = d} = 3\,m$!

Problems II

8) A **proton** synchrotron has a magnetic field of **1.5T** rising linearly in **1 s** from **1** to **10 GeV**. What is the momentum at injection and top energy and what the magnetic rigidity? If **2/3** of the circumference are bending magnets what is the bending radius and the mean radius? What is the revolution frequency at injection and top energy? Taking the revolution frequency at injection, calculate the voltage per turn. If the synchronous phase is **45^{o}**, what is the peak voltage in the cavity?

$T_{inj} = 1\ \text{GeV} \rightarrow E_{inj} = 1.938\ \text{GeV}$

$T_e \quad 10\ \text{GeV} \rightarrow E_{top} = 10.938\ \text{GeV}.$

$$E^2 = (m_0c^2)^2 + (pc)^2 \rightarrow pc = \begin{cases} \xrightarrow{inj} 1.696\ \text{GeV} \\ \xrightarrow{top} 10.89\ \text{GeV} \end{cases}$$

$$B\rho = 3.3356\, p = \begin{cases} \xrightarrow{inj} 5.66\ \text{Tm} \\ \xrightarrow{top} 36.4\ \text{Tm} \end{cases}$$

$$\rho = \frac{B\rho}{B_{top}} = 24.23\ \text{m}$$

The mean radius

$$\bar{R} = \frac{3}{2}\rho = 36.35\ \text{m}$$

$$\beta = \frac{pc}{E} \begin{cases} \rightarrow 0.875 \text{ at injection} \\ \rightarrow 0.9963 \text{ at top energy} \end{cases}$$

Problems II

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$$f_{rev} = \frac{v}{2\pi R} = \frac{\beta c}{2\pi R} = \begin{cases} 1.148 \text{ MHz at injection} \\ 1.308 \text{ MHz at top energy} \end{cases}$$

$$f_{rev} V = \Delta E^{=9\,\text{GeV}} \rightsquigarrow V = 7.84 \text{ kV}$$

The peak voltage is $V^{P} \cdot \sin 45 = V \rightarrow$

$$\underline{V^{P} = 11.04 \text{ kV}}$$

Problems II

9) For the same ring with a mean **dispersion** of **9m** what is a) the transition energy (b) momentum at transition (c) the slippage factor at injection and top energy (d) If the **harmonic number** is **10** what is the synchrotron frequency at injection and top energy?

(4.8) a) $\frac{1}{\gamma_{tr}^2} = \left\langle \frac{D}{\rho} \right\rangle \Rightarrow \frac{\bar{D}}{\bar{R}} \gamma_{tr} \simeq$ 2

$E^{tr} = \gamma^{tr} \cdot E_0 = 1.886$ GeV.

b) $pc^{tr} = \sqrt{E^{tr\,2} - E_0^2} =$ 1.64 GeV

c) At injection $\gamma = \frac{E^{inj}}{E_0} = 2.066$

At top energy $\gamma = \frac{E^{top}}{E_0} = 11.658$

$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma^{tr\,2}}$

inj $\rightarrow$ 0.013

top $\rightarrow$ -0.24

d) $f_s = \sqrt{\frac{|\eta| e V_0}{2\pi E_0 \beta^2 \gamma h}}\, f_{RF}$

$f_{RF} \rightarrow h \cdot f_{rev.}$

inj $\rightarrow$ 438 Hz

top $\rightarrow$ 65 Hz

Problems II

10) Consider a **400 GeV** proton synchrotron with **108 3.22 m-long** focusing and defocusing quads of **19.4 T/m**, with a horizontal and vertical **beta** of **108 m** and **18 m** in the focusing quads which is inversed for the defocusing ones. What is the chromaticity of the machine?

- The magnetic rigidity is $B\rho \;\; [\mathrm{T\ m}] = \dfrac{1}{0.2998}\beta_r E \;\; [\mathrm{GeV}]$
- For 400GeV, the relativistic beta is almost 1 and then the magnetic rigidity is

$$B\rho = 1334 \;\; \mathrm{T\ m}$$

- The focusing normalized gradient is $K_F = \dfrac{G_F}{B\rho} = \dfrac{19.4}{1334} = 0.015\ \mathrm{m}^{-2}$
- The defocusing one is just the same with opposite sign $K_D = -0.015\ \mathrm{m}^{-2}$
- The chromaticity of the machine is $\xi_{x,y} = -\dfrac{1}{4\pi}\sum_i \beta^i_{x,y} K^i_{x,y}$
- By splitting again the focusing and defocusing quads' contribution, we have

$$\xi_{x,y} = -\frac{1}{4\pi} NlK(\pm\beta^F_{x,y} \mp \beta^D_{x,y})$$

- This gives in both planes $\xi_{x,y} = -\dfrac{108 \times 3.22 \times 0.015}{4\pi}(108 - 18) = -36.2$