Εισαγωγή στη Φυσική των Επιταχυντών

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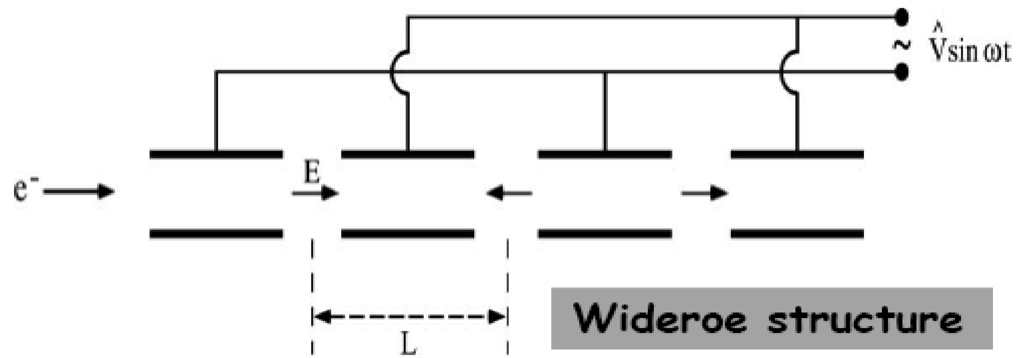
Longitudinal dynamics

- RF acceleration
- Energy gain and phase stability
- Momentum compaction and transition
- Equations of motion
  - Small amplitudes
  - Longitudinal invariant
- Separatrix
- Energy acceptance
- Stationary bucket
- Adiabatic damping

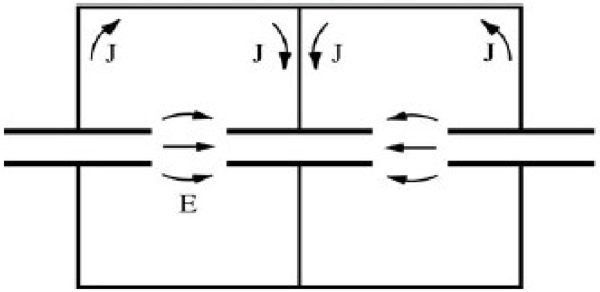
RF acceleration

- The use of RF fields allows an arbitrary number of accelerating steps in gaps and electrodes fed by RF generator
- The electric field is not longer continuous but sinusoidal alternating half periods of acceleration and deceleration
- The synchronism condition for RF period T_{RF} and particle velocity v

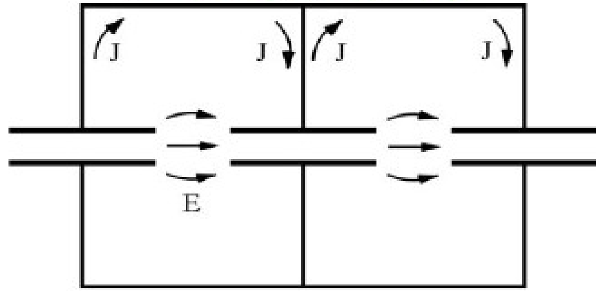
$$L = vT_{RF}/2 = \beta c \frac{\pi}{\omega_{RF}} = \beta\lambda/2$$



L = vT/2 (π mode)



L = vT (2π mode)



Energy gain

Assuming a sinusoidal electric field $E_z = E_0 \cos(\omega_{RF} t + \phi_s)$ where the synchronous particle passes at the middle of the gap *g*, at time *t* = *0*, the energy is

$$W(r,t) = q\int E_z dz = q\int_{-g/2}^{g/2} E_0 \cos(\omega_{RF}\frac{z}{v} + \phi_s)\, dz$$

And the energy gain is $\Delta W = qE_0 \int_{-q/2}^{g/2} \cos(\omega_{RF}\frac{z}{v}) dz$

and finally $\Delta W = qV \frac{\sin\Theta/2}{\Theta/2} = qVT$ with the transit time factor defined as

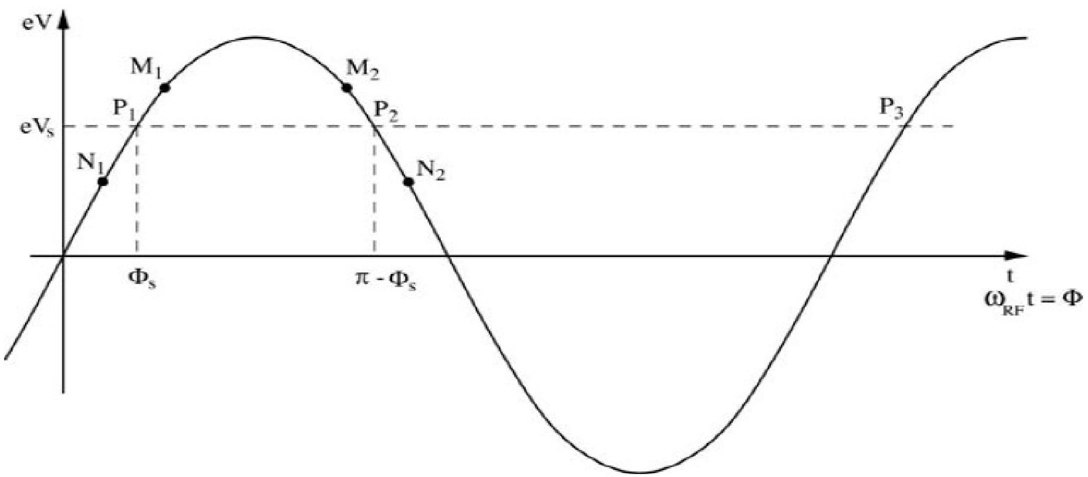
$$T = \frac{\sin(\omega g/2v)}{\omega g/2v}$$

It can be shown that in general

$$T = \frac{\int_{-g/2}^{g/2} E(0,z)\cos\omega t(z)\, dz}{\int_{-g/2}^{g/2} E(0,z)\, dz}$$

Phase stability

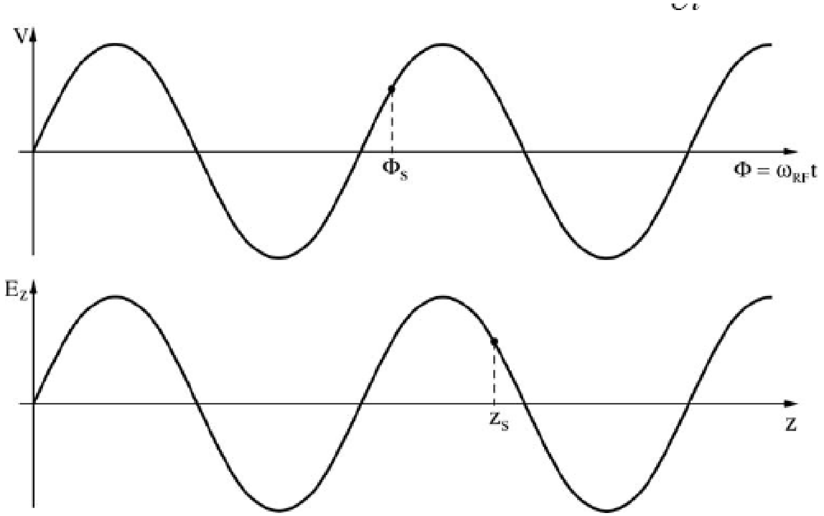
- Assume that a synchronicity condition is fulfilled at the phase ϕ_s and that energy increase produces a velocity increase
- Around point P_1, that arrives earlier (N_1) experiences a smaller accelerating field and slows down
- Particles arriving later (M1) will be accelerated more
- A restoring force that keeps particles oscillating around a stable phase called the synchronous phase ϕ_s
- The opposite happens around point P2 at π-$\phi_{s,}$ i.e. M2 and N2 will further separate



RF de-focusing

In order to have stability, the time derivative of the Voltage and the spatial derivative of the electric field should satisfy

$$\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E}{\partial z} < 0$$



In the absence of electric charge the field is given by M

$$\nabla \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial E_x}{\partial x} > 0$$

where x represents the generic transverse direction.

External focusing is required by using quadrupoles or solenoids

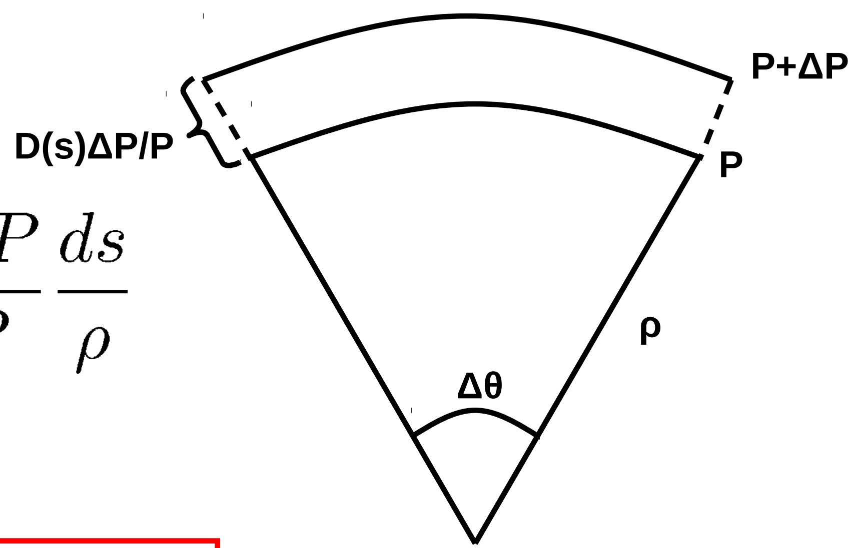
Momentum compaction

- Off-momentum particles on the dispersion orbit travel in a different path length than on-momentum particles
- The change of the path length with respect to the momentum spread is called **momentum compaction**

$$\alpha_c = \frac{\Delta C}{C} / \frac{\Delta P}{P}$$

- The change of circumference is

$$\Delta C = \oint D \frac{\Delta P}{P} d\theta = \oint D \frac{\Delta P}{P} \frac{ds}{\rho}$$



- So the momentum compaction is

$$\boxed{\alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds = \left\langle \frac{D(s)}{\rho(s)} \right\rangle}$$

Transition energy

- The revolution frequency of a particle is $f=\frac{v}{2\pi\rho}=\frac{\beta c}{2\pi\rho}$
- The change in frequency is $\frac{\Delta f}{f}=\frac{\Delta\rho}{\rho}-\frac{\Delta\beta}{\beta}$
- From the relativistic momentum $Pc=\beta E$ e have

$$\frac{\Delta P}{P}=\frac{\Delta\beta}{\beta}+\frac{\Delta E}{E}$$ for wl $\beta^2\frac{\Delta P}{P}$

$$\frac{\Delta\beta}{\beta}=\frac{1}{\gamma^2}\frac{\Delta P}{P}$$

and the revolution frequency $\frac{\Delta f}{f}=(\frac{1}{\gamma^2}-\alpha_c)\frac{\Delta P}{P}$

The slippage factor is given by $\eta=\frac{1}{\gamma^2}-\alpha_c$

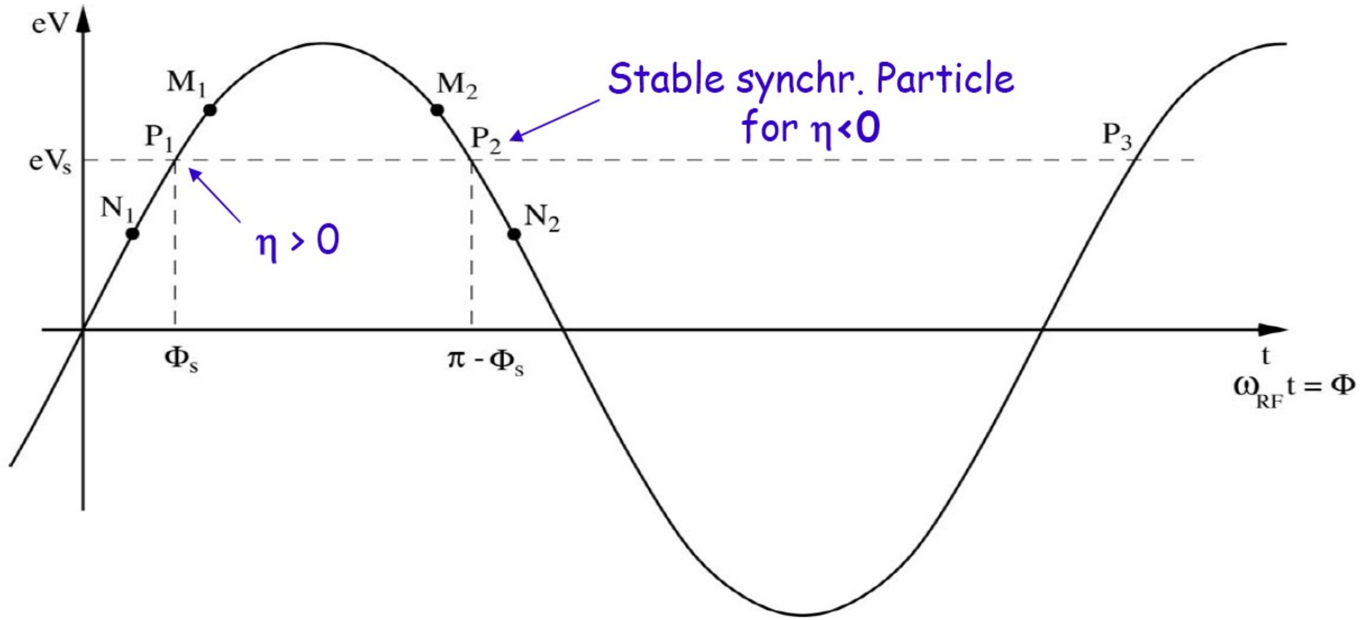
For vanishing slippage factor, the transition energy is defined $\boxed{\gamma_t=\frac{1}{\sqrt{\alpha_c}}}$

Synchrotron

- Frequency modulated but also *B*-field increased **synchronously** to match energy and keep revolution radius constant.
- The number of stable synchronous particles is equal t the harmonic number *h*. They are equally spaced along the circumference.
- Each synchronous particle has the nominal energy and follow the nominal trajectory
- Magnetic field increases with momentum and the per turn change of the momentum is

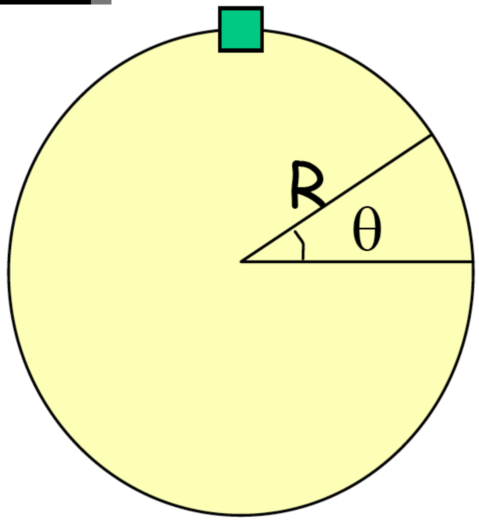
$$(\Delta p)_{turn} = e\rho\, B' T_r = \frac{2\pi\, e\rho\, RB'}{v}$$

Phase stability on electron synchrotrons



- For electron synchrotrons, the relativistic γ is very large and
  $$\eta = \frac{1}{\gamma^2} - \alpha_c \approx -\alpha_c < 0$$
  m compaction is
- Above transition, an increase in energy is followed by lower revolution frequency
- A delayed particle with respect to the synchronous one will get closer to it (gets a smaller energy increase) and phase stability occurs at the point P2 (π - ϕ_s)

Energy and phase relation



- The RF frequency and phase are related to the revolution ones as follows

$$f_{RF} = hf_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

and $\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$

- From the definition of the momentum compaction and for electrons

$$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s = \frac{E_s}{\omega_{rs}}\left(\frac{d\omega_r}{dE}\right)_s \cong -\alpha_c$$

- Replacing the revolution frequency change, the following relation is obtained between the energy and the RF phase time derivative

$$\frac{\Delta E}{E_s} = \frac{1}{\omega_{rs}\alpha_c h}\frac{d\phi}{dt} = \frac{R}{c\alpha_c h}\dot{\phi}$$

Longitudinal equations of motion

- The energy gain per turn with respect to the energy gain of the synchronous particle is

$$(\Delta E)_{turn} = e\hat{V}(\sin\phi - \sin\phi_s)$$

- The rate of energy change can be approximated by

$$\frac{d(\Delta E)}{dt} \cong (\Delta E)_{turn} f_{rs} = \frac{c}{2\pi R} e\hat{V}(\sin\phi - \sin\phi_s)$$

- The second energy phase relation is written as

$$\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right) = \frac{ce\hat{V}}{2\pi R E_s}(\sin\phi - \sin\phi_s)$$

- By combining the two energy/phase relations, a 2nd order differential equation is obtained, similar the pendulum

$$\frac{d}{dt}\left(\frac{R}{c\alpha_c h}\frac{d\phi}{dt}\right) + \frac{ce\hat{V}}{2\pi R E_s}(\sin\phi - \sin\phi_s) = 0$$

Small amplitude oscillations

- Expanding the harmonic functions in the vicinity of the synchronous phase

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

- Considering also that the coefficient of the phase derivative does not change with time, the differential equation reduces to one describing an harmonic oscillator

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$

with frequency

$$\Omega_s^2 = -\frac{c^2 e \alpha_c h \hat{V} \cos\phi_s}{R^2 2\pi E_s}$$

- For stability, the square of the frequency should positive and real, which gives the same relation for phase stability when particles are above transition

$$\cos\phi_s < 0 \Rightarrow \pi/2 < \phi_s < \pi$$

Longitudinal motion invariant

- For large amplitude oscillations the differential equation of the phase is written as

$$\ddot{\phi}+\frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi-\sin\phi_s\right)=0$$

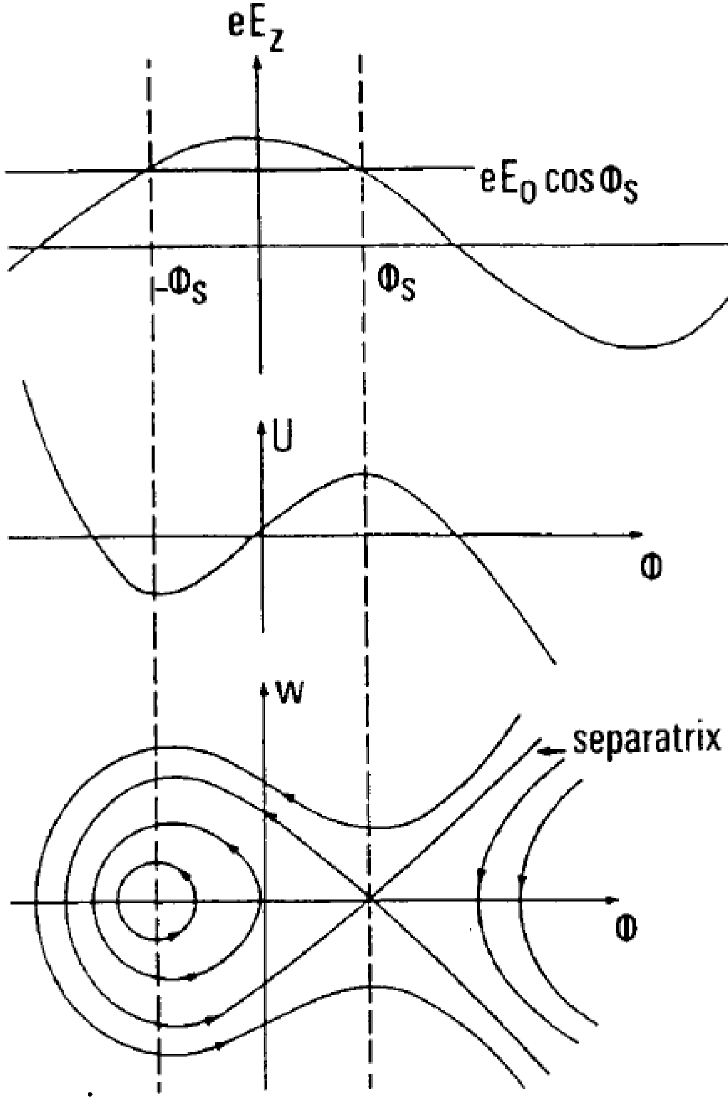
- Multiplying by the time derivative of the phase and integrating, an invariant of motion is obtained

$$\frac{\dot{\phi}^2}{2}-\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi+\phi\sin\phi_s\right)=I$$

reducing to the following expression, for small amplitude oscillations

$$\frac{\dot{\phi}^2}{2}+\frac{\Omega_s^2}{2}\Delta\phi=I$$

Separatrix



- In the phase space (energy change versus phase), the motion is described by distorted circles in the vicinity of ϕ_s (stable fixed point)
- For phases beyond π - ϕ_s (unstable fixed point) the motion is unbounded in the phase variable, as for the rotations of a pendulum
- The curve passing through π - ϕ_s is called the **separatrix** and the enclosed area **bucket**

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s\right)$$

Energy acceptance

- The time derivative of the RF phase (or the energy change) reaches a maximum (the second derivative is zero) at the synchronous phase
- The equation of the separatrix at this point becomes

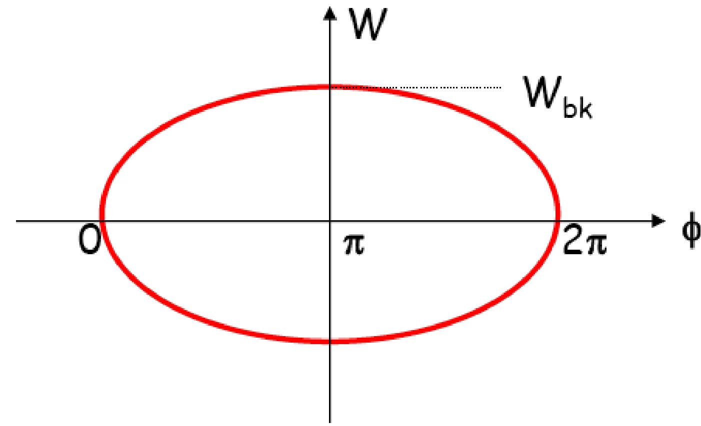
$$\dot{\phi}^2_{\max} = 2\Omega_s^2\left(2 + (2\phi_s - \pi)\tan\phi_s\right)$$

- Replacing the time derivative of the phase from the first energy phase relation

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \mp\sqrt{\frac{q\hat{V}}{\pi h \alpha_c E_s}\left(2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right)}$$

- This equation defines the energy acceptance which depends strongly on the choice of the synchronous phase. It plays an important role on injection matching and influences strongly the electron storage ring lifetime

Stationary bucket



- When the synchronous phase is equal to 0 (below transition) or π (above transition), there is no acceleration. The equation of the separatrix is written

$$\frac{\dot{\phi}^2}{2} = 2\Omega_s^2 \sin^2\frac{\phi}{2}$$

- Using the (canonical) variable $W = 2\pi\frac{\Delta E}{\omega_{rs}} = 2\pi\frac{E_s R}{h\,\alpha_c\,\omega_{rs}}\dot{\phi}$ and replacing the expression for the synchrotron frequency

$$W = \pm 2\frac{C}{c}\sqrt{\frac{q\hat{V}E_s}{2\pi\,h\alpha_c}}\cdot\sin\frac{\phi}{2}$$

For $\phi = \pi$, the bucket **height** is

$$W_{bk} = 2\frac{C}{c}\sqrt{\frac{e\hat{V}E_s}{2\pi\,h\alpha_c}}$$

and the **area**

$$A_{bk} = 2\int_0^{2\pi} W\,d\phi = 8W_{bk}$$

Adiabatic damping

- The longitudinal oscillations can be damped directly by acceleration itself. Consider the equation of motion when the energy of the synchronous particle is not constant

$$\frac{d}{dt}\left(E_s\dot{\phi}\right) = -\Omega_s^2 E_s \Delta\phi$$

- From this equation, we obtain a 2nd order differential equation with a damping term

$$\ddot{\phi} + \frac{\dot{E}_s}{E_s}\dot{\phi} + \Omega_s^2 \Delta\phi = 0$$

- From the definition of the synchrotron frequency the damping coefficient is

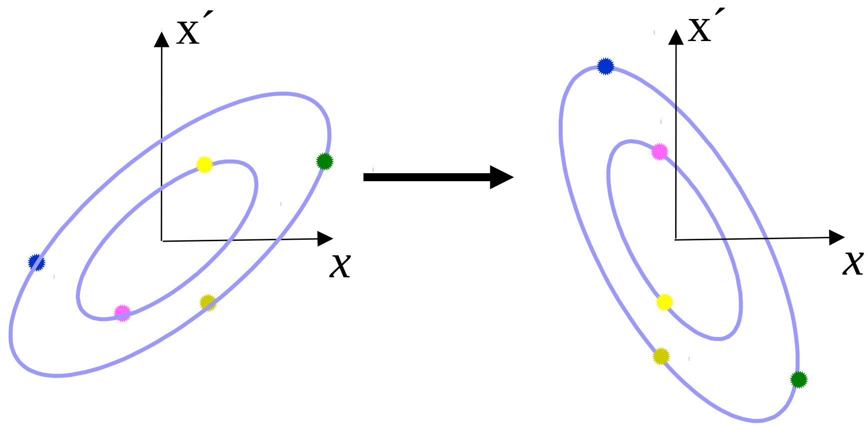
$$\frac{\dot{E}_s}{E_s} = -2\frac{\dot{\Omega}_s}{\Omega_s}$$

Outline – Phase space concepts

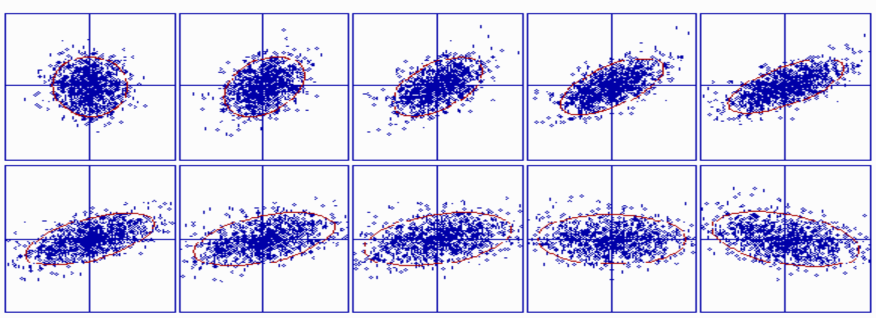
- Transverse phase space and Beam representation
- Beam emittance
- Liouville and normalised emittance
- Beam matrix
- RMS emittance
- Betatron functions revisited
- Gaussian distribution

Transverse Phase Space

- Under linear forces, any particle moves on ellipse in phase space (x,x’), (y,y’).
- Ellipse rotates and moves between magnets, but its area is preserved.
- The area of the ellipse defines the **emittance**

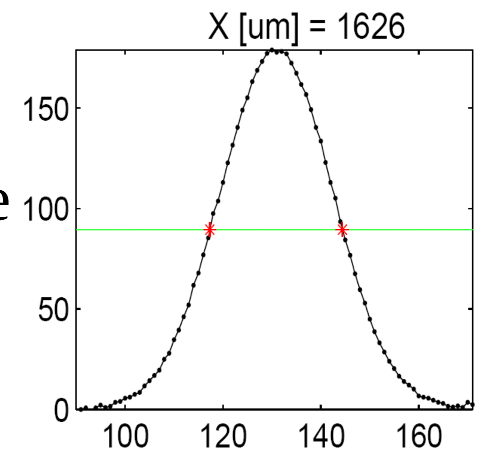


- The equation of the ellipse is
$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \epsilon$$
with α,β,γ, the twiss parameters
- Due to large number of particles, need of a statistical description of the beam, and its size

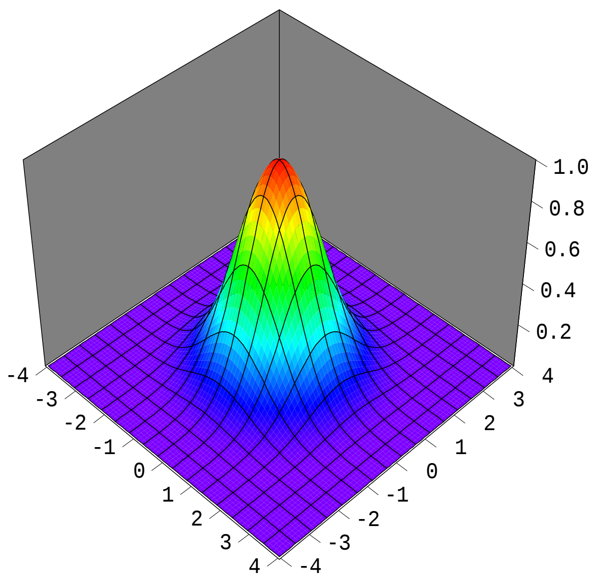


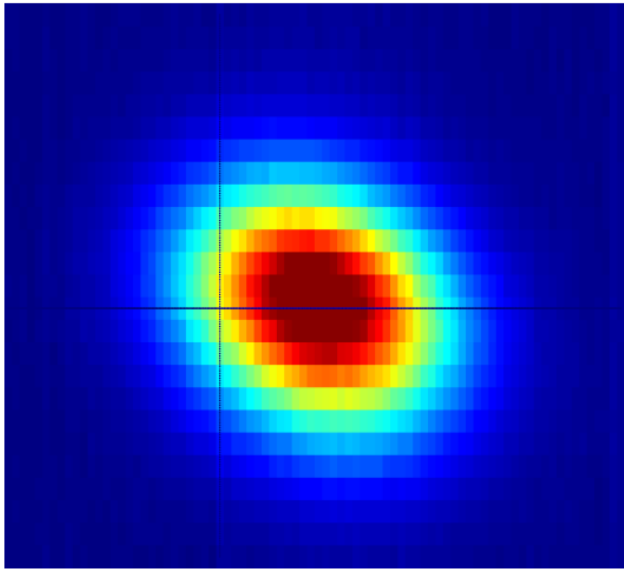
Beam representation

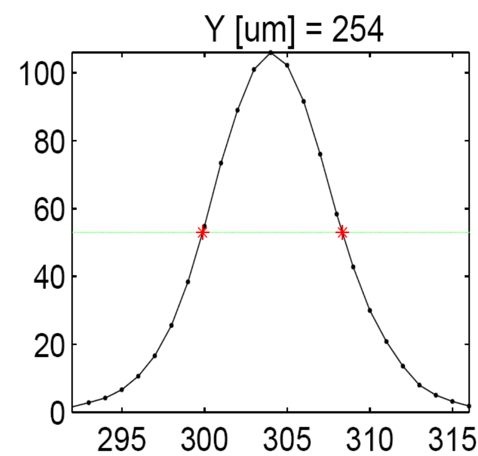
- Beam is a set of millions/billions of particles (N)
- A macro-particle representation models beam as a set of n particles with n<<N
- Distribution function is a statistical function representing the number of particles in phase space between $\mathbf{u} + d\mathbf{u}$, $\mathbf{u}' + d\mathbf{u}'$



$$f(\mathbf{u}, \mathbf{u}')d\mathbf{u}d\mathbf{u}' = \text{number of particles}$$







Liouville emittance

- Emittance represents the phase-space volume occupied by the beam
- The phase space can have different dimensions
  - 2D **(x, x’)** or **(y, y’)** or **(φ, E)**
  - 4D **(x, x’,y, y’)** or **(x, x’, φ, E)** or **(y, y’, φ, E)**
  - 6D **(x, x’, y, y’, φ, E)**
- The resolution of my beam observation is very large compared to the average distance between particles.
- The beam modeled by phase space **distribution function**

$$f(x, x', y, y', \phi, E)$$

- The volume of this function on phase space is the beam **Liouville emittance**

Vlasov and Boltzmann equations

- The evolution of the distribution function is described by **Vlasov** equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{\gamma m_0}\frac{\partial f}{\partial \mathbf{q}} + \mathbf{F}(\mathbf{q})\frac{\partial f}{\partial \mathbf{p}} = 0$$

- Mathematical representation of **Liouville theorem** stating the conservation of phase space volume $(\mathbf{q}, \mathbf{p})$
- In the presence of fluctuations (**radiation**, collisions, etc.) distribution function evolution described by **Boltzmann equation**

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{\gamma m_0}\frac{\partial f}{\partial \mathbf{q}} + \mathbf{F}(\mathbf{q})\frac{\partial f}{\partial \mathbf{p}} = \left.\frac{df}{dt}\right|_{\text{fluct}}$$

- The distribution evolves towards a **Maxwell-Boltzmann statistical equilibrium**

2D and normalized emittance

- When motion is uncoupled, Vlasov equation still holds for each plane individually

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{p_u}{\gamma m_0}\frac{\partial f}{\partial u} + \mathbf{F}(u)\frac{\partial f}{\partial p} = 0$$

- The Liouville emittance in the 2D (u, p_u) phase space is still conserved
- In the case of acceleration, the emittance is conserved in the (u, p_u) but not in the (u, u') **diabatic damping)**
- Considering that

$$u' = \frac{du}{ds} = \frac{p_u}{p_s}$$

the beam is conserved in the phase space $(u, u' p_s)$

- Define a **normalised emittance which is conserved during acceleration**

$$\epsilon_n = \beta_r \gamma_r \epsilon$$

Beam matrix

- We would like to determine the transformation of the beam enclosed by an ellipse through the accelerator
- Consider a vector **u = (x,x',y,y',…)** in a generalized n-dimensional phase space. In that case the ellipse transformation is

$$\mathbf{u}^T \cdot \mathbf{\Sigma}^{-1} \cdot \mathbf{u} = \mathcal{I}$$

- Application to one dimension gives $\Sigma_{11}u^2 + 2\Sigma_{22}uu' + \Sigma_{22}u'^2 = 1$ and comparing with $\gamma_u u^2 + 2\alpha_u uu' + \beta_u u'^2 = \epsilon_u$ provides the beam matrix $\Sigma_u = \begin{pmatrix} \beta_u & -\alpha_u \\ -\alpha_u & \gamma_u \end{pmatrix} \epsilon_u = \mathcal{B}\epsilon_u$ which can be expanded to more dimensions
- Evolution of the n-dimensional phase space from position 1 to position 2, through transport matrix $\mathcal{M}$

$$\mathcal{M} \cdot \mathbf{\Sigma}_1 \cdot \mathcal{M}^T = \mathbf{\Sigma}_2$$

Root Mean Square (RMS) beam parameters

- The average of a function on the beam distribution defined

$$\langle g(\mathbf{u},\mathbf{u}')\rangle = \frac{1}{n}\sum_{i=1}^{n} g(u_i,u_i') = \frac{1}{N}\iint f(\mathbf{u},\mathbf{u}')g(\mathbf{u},\mathbf{u}')d\mathbf{u}d\mathbf{u}'$$

- Taking the square root, the following **Root Mean Square (RMS)** quantities are defined
  - **RMS beam size**

$$u_{\mathrm{rms}} = \sqrt{\sigma_u} = \sqrt{\langle (u - \langle u\rangle)^2\rangle}$$

  - **RMS beam divergence**

$$u'_{\mathrm{rms}} = \sqrt{\sigma'_u} = \sqrt{\langle (u' - \langle u'\rangle)^2\rangle}$$

  - **RMS coupling**

$$(uu')_{\mathrm{rms}} = \sqrt{\sigma_{uu'}} = \sqrt{\langle (u - \langle u\rangle)(u' - \langle u'\rangle)\rangle}$$

RMS emittance

- Beam modelled as macro-particles
- Involved in processed linked to the statistical size
- The **rms emittance** is defined as

$$\epsilon_{\mathrm{rms}} = \sqrt{\langle u\rangle^2 \langle u'\rangle^2 - \langle uu'\rangle^2}$$

- It is a statistical quantity giving information about the minimum beam size
- For linear forces the rms emittance is conserved in the case of linear forces
- The determinant of the rms beam matrix $det(\Sigma_{\mathrm{rms}}) = \epsilon_{\mathrm{rms}}$
- Including acceleration, the determinant of 6D transport matrices is not equal to 1 but

$$\det(\mathcal{M}_{1\rightarrow 2}) = \sqrt{\frac{\beta_{r2}\gamma_{r2}}{\beta_{r1}\gamma_{r1}}}$$

Beam betatron functions

- The best ellipse fitting the beam distribution is

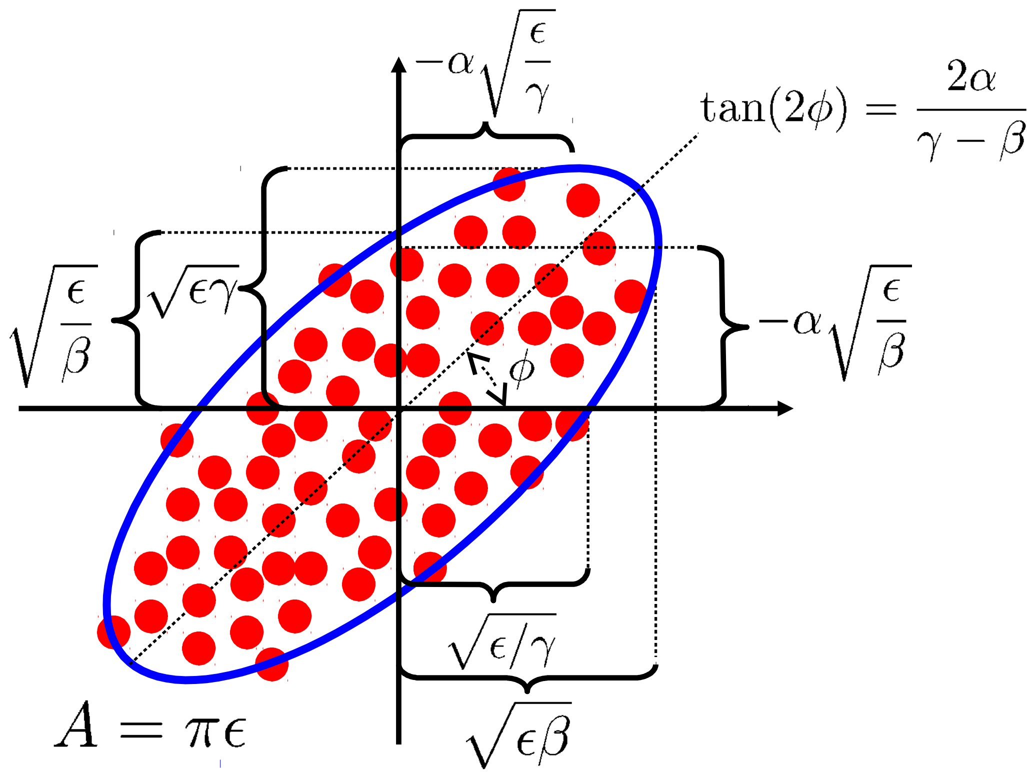
$$\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = \epsilon_u$$

- The beam betatron functions can be defined through the rms emittance

$$\beta_u = \frac{u^2_{\rm rms}}{\epsilon_{\rm rms}} = \frac{\sigma_u}{\epsilon_{\rm rms}}$$

$$\gamma_u = \frac{u'^2_{\rm rms}}{\epsilon_{\rm rms}} = \frac{\sigma'_u}{\epsilon_{\rm rms}}$$

$$\alpha_u = \frac{(u\ u')_{\rm rms}}{\epsilon_{\rm rms}} = \frac{\sigma_{uu'}}{\epsilon_{\rm rms}}$$



Gaussian distribution

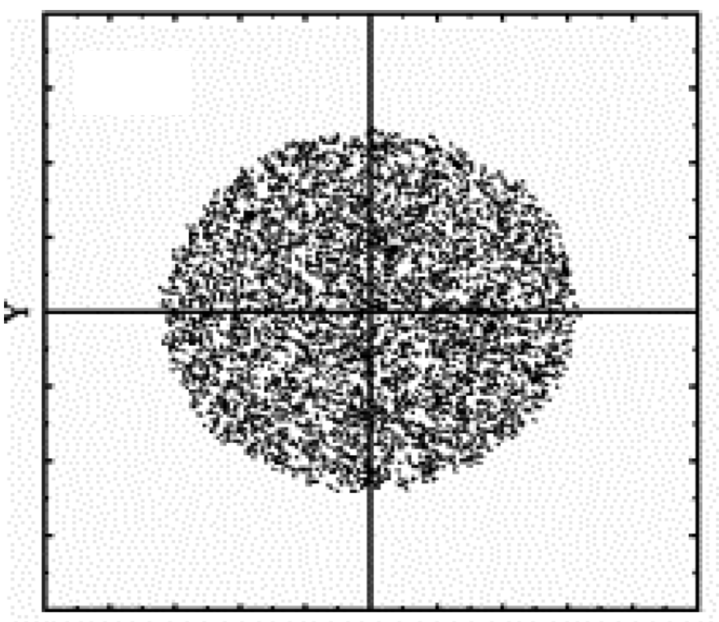
- The **Gaussian distribution** has a gaussian density profile in phase space

$$f(x,x',y,y') = \frac{N}{A} exp\left(-\frac{\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2}{2\epsilon_{x,\mathrm{rms}}} + \frac{\gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2}{2\epsilon_{y,\mathrm{rms}}}\right)$$

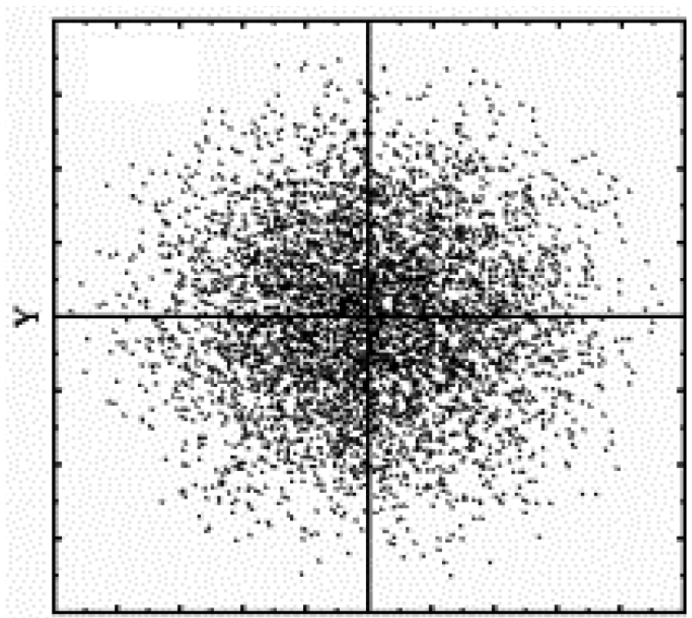
for which $\int f(\mathbf{u},\mathbf{u}')d\mathbf{u}d\mathbf{u}' = N$

- The beam boundary is $\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = n^2 \epsilon_{u,\mathrm{rms}}$

Uniform (KV)



Gaussian

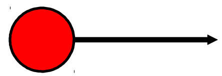


Outline

- Radiation damping
  - Synchrotron oscillations
  - Betatron oscillations
  - Robinson theorem
- Radiation integrals
- Quantum excitation
- Equilibrium emittances

- Up to this point, the transport of a relativistic particle around a ring was treated as a conservative process
- The particle change of momentum (acceleration) results in emission of synchrotron radiation
- It turns out that this is much more important in circular then linear accelerators
- The emission of synchrotron radiation results in energy lost by the particle and the damping of oscillations, called **radiation damping**
- This energy lost is recovered by the RF accelerating cavities in the longitudinal direction but not in the transverse

Why circular machines?



$$\mathbf{p} = m_0 \mathbf{v} \qquad v << c$$

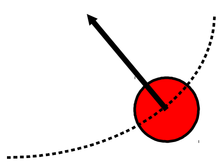
$$P_s = \frac{e^2}{6\pi\varepsilon_0 m_0^2 c^3} \left(\frac{d\mathbf{p}}{dt} \right)^2$$

Larmor Power radiated by non-relativistic particles is very small

$$\mathbf{p} = \gamma m_0 \mathbf{v} \qquad v \approx c$$

$$P_s = \frac{e^2}{6\pi\varepsilon_0 m_0^2 c^3} \left(\frac{dp}{dt} \right)^2$$

Power radiated by relativistic particles in linear accelerators is negligible



$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho^2}$$

Power radiated by relativistic particles in circular accelerators is very strong (Liénard, 1898)

Lienard's Paper

- "Electric and Magnetic Field produced by an electric charge concentrated at a point and travelling on an arbitrary path"

Prophetically published in the french journal "The Electric Light"

L'Éclairage Électrique

REVUE HEBDOMADAIRE D'ÉLECTRICITÉ

CHAMP ÉLECTRIQUE ET MAGNÉTIQUE

PRODUIT PAR UNE CHARGE ÉLECTRIQUE CONCENTRÉE EN UN POINT ET ANIMÉE D'UN MOUVEMENT QUELCONQUE

Admettons qu'une masse électrique en mouvement de densité ρ et de vitesse u en chaque point produit le même champ qu'un courant de conduction d'intensité $u\rho$. En conservant les notations d'un précédent article (1) nous obtiendrons pour déterminer le champ, les équations

avec les analogues déduites par permutation tournante et en outre les suivantes

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0 \qquad (4)$$

De ce système d'équations on déduit facilement les relations

(1) La théorie de Lorentz, *L'Éclairage Électrique*, t. XIV, p. 417. α, β, γ sont les composantes de la force magnétique et f, g, h, celles du *déplacement* dans l'éther.

Soient maintenant quatre fonctions ψ, F, G, H définies par les conditions

On satisfera aux conditions (5) et (6) en prenant

$$4\pi f = -\frac{d\psi}{dx} - \frac{1}{V^2}\frac{dF}{dt} \qquad (9)$$

$$\alpha = \frac{dH}{dy} - \frac{dG}{dz} \qquad (10)$$

Quant aux équations (1) à (4), pour qu'elles soient satisfaites, il faudra que, en plus de (7) et (8), on ait la condition

$$\frac{d\psi}{dt} + \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0. \qquad (11)$$

Occupons-nous d'abord de l'équation (7). On sait que la solution la plus générale est la suivante :

Why electrons?

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho^2}$$

$$\Delta E = \frac{e^2}{3\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho}$$

Power inversely proportional to 4th power of rest mass (proton **2000 times** heavier than electron) On the other hand, for multi TeV hadron colliders (LHC, FCCpp) synchrotron radiation is an important issue (protection with beam screens)

By integrating around one revolution, the **energy loss per turn** is obtained. For the ILC DR, it is around **4.5 MeV/turn**. On the other hand, for LEPII (**120 GeV**) it was **6 GeV/turn**, or for FCCee (ttbar flavor at **175 GeV**), it will be **7.5GeV/turn** i.e. circular electron/positron machines of hundreds of GeV become quite demanding with respect to RF power (and extremely long)

Synchrotron Radiation formulas for e^-/e^+

- The power radiated by a relativistic electron can be rewritten as

$$\boxed{P_\gamma = \frac{cC_\gamma E^4}{2\pi\rho^2} = \frac{e^2c^3}{2\pi}C_\gamma E^2 B^2}$$

with

$$C_\gamma = 8.85 \times 10^{-5} \frac{\mathrm{m}}{(\mathrm{GeV})^3}$$

- The energy loss per turn can be expressed as

$$\boxed{U_0 = \frac{C_\gamma E^4}{2\pi} \oint \frac{ds}{\rho^2}}$$

with

$$\mathcal{I}_2 = \oint \frac{ds}{\rho^2}$$

the 2nd **radiation integral**

- For a lattice with uniform bending radius (iso-magnetic) this yields

$$\boxed{U_0[keV] = 88.5 \frac{E^4}{\rho} \frac{[\mathrm{GeV}]^4}{[\mathrm{m}]}}$$

- If this energy were not recovered, particles would gradually spiral inward until lost on vacuum chamber wall
- RF cavities replace this lost energy by providing momentum kicks to the beam in the longitudinal direction

Damping of synchrotron oscillations

- Consider the differential equation of the energy for longitudinal motion

$$\Delta\ddot{E} + \alpha_s \Delta\dot{E} + \Omega^2 \Delta E = 0$$

with damping coefficient $\alpha_s = \frac{1}{2T_0}\frac{dU}{dE}$

where U is the energy requirement per turn of the particle, and T_0 the revolution period and the synchrotron frequency

$$\Omega^2 = \frac{e\alpha_c \omega_{RF} V_0 \cos\phi_s}{ET_0}$$

- The solution can be written as a damped oscillation in energy and time with respect to the ideal synchromous particle

$$\Delta E(t) = A_E e^{-\alpha_s t}\cos(\Omega t - \phi_s)$$

$$\tau(t) = -\frac{\alpha_c A_E}{E_0 \Omega} e^{-\alpha_s t}\sin(\Omega t - \phi_s)$$

Εισαγωγή στη Φυσική των Επιταχυντών

Damping of synchrotron oscillations

- Note that the synchrotron motion is damped towards the motion of the synchronous particle
- The damping coefficient is dependent on the energy of the particle through the radiated power but also through the revolution period. In the following, we try to establish this relationship
- A particle with energy spread follows a dispersive trajectory with dispersion *D*

$$ds' = (1+\frac{\Delta x}{\rho})ds = (1+\frac{D}{\rho}\frac{\Delta E}{E})ds$$

- The energy req d by the integral of the radiated power in one revolution

$$U = \oint P_s dt = \oint P_s ds'/c = \frac{1}{c}\oint P_s(1+\frac{D}{\rho}\frac{\Delta E}{E})ds$$

- Differer

$$\frac{dU}{dE} = \frac{1}{c}\oint\left[\frac{dP_s}{dE} + \frac{D}{\rho}(\frac{dP_s}{dE}\frac{\Delta E}{E} + \frac{P_s}{E})\right]ds$$

- Taking into account that the average energy spread around the ring should be zero the previous integral is written:

$$\frac{dU}{dE} = \frac{1}{c}\oint\left(\frac{dP_s}{dE} + \frac{D}{\rho}\frac{P_s}{E}\right)ds$$

- Setting $\mathcal{C} = \frac{e^4 c^3}{6\pi\varepsilon_0 (m_0 c^2)^4}$ taking into account the definition of the magnetic rigidity, the expression of the radiation power is written $P_s = \mathcal{C}E^2 B^2$
- Its derivative with respect to the energy gives

$$\frac{dP_s}{dE} = 2\frac{P_s}{E}(1 + Dk\rho)$$

where we used the identity $\frac{dB}{dE} = \frac{dB}{dx}\frac{dx}{dE} = \frac{dB}{dx}\frac{D}{E} = Bk\rho\frac{D}{E}$

- Replacing in the integral at the top,

$$\frac{dU}{dE} = \frac{2U_0}{E} + \frac{1}{cE}\oint DP_s\left(2k\rho + \frac{1}{\rho}\right)ds$$

Damping of synchrotron oscillations III

- Replacing in the last integral the expression of the power

$$\oint DP_s \left(2k\rho + \frac{1}{\rho}\right) ds = \frac{\mathcal{C}E^4}{e^2c^2} \oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right) ds$$

and taking into account that $U_0 = \frac{1}{c}\oint P_s ds = \frac{\mathcal{C}E^4}{e^2c^3}\oint \frac{ds}{\rho^2}$

the damping of synchrotron motion is written

$$\alpha_s = \frac{1}{2T_0}\frac{dU}{dE} = \frac{U_0}{2ET_0}(2+\mathcal{D}) = \frac{U_0}{2ET_0}\mathcal{J}_s$$

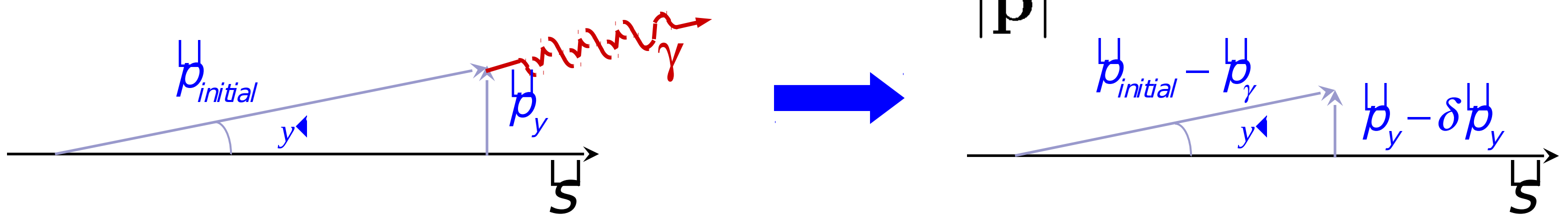
with the damping partition number defined as

$$\mathcal{D} = \frac{\oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right) ds}{\oint \frac{ds}{\rho^2}} = \frac{\mathcal{I}_4}{\mathcal{I}_2}$$

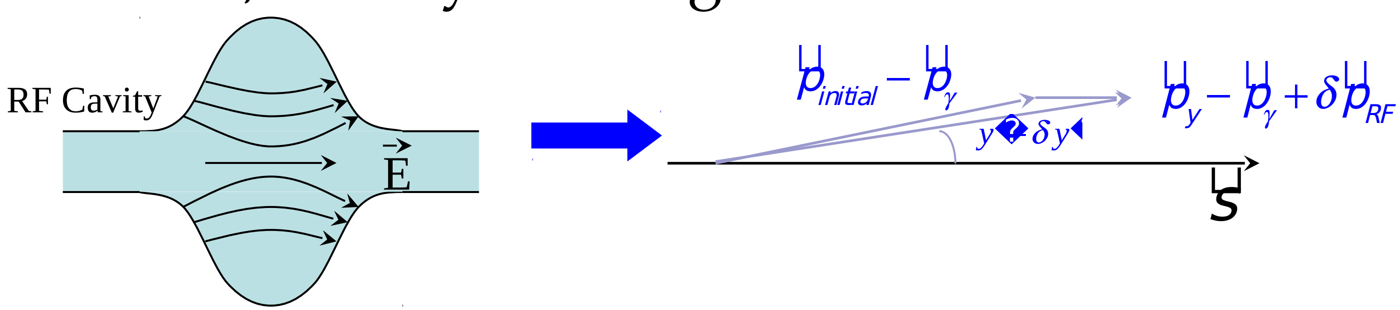
- This is a parameters entirely defined by the lattice!
- Bending magnets and quads are usually separated and the damping partition number is usually extremely small

Damping of vertical oscillations

- Synchrotron radiation emitted in the direction of motion of electron, whose momentum is reduced
- This reduces the vertical component of the momentum but the angle remains the same $y' = \dfrac{\delta p_\perp}{|\mathbf{p}|}$



- The key for betatron damping is the energy recovery by the RF cavities, as only the longitudinal momentum is restored



- The change in energy will not affect the vertical position but the angle changes proportionally $\delta y' = y' \dfrac{\delta E}{E}$

Damping of vertical oscillations II

- Recall now solution of Hill's equations in the vertical plane, assuming that the beta function is slowly varying (i.e. alpha function is zero), for simplicity

$$y = A\cos\phi \;\text{ and }\; y' = -\frac{A}{\beta(s)}\sin\phi$$

- The betatron oscillation amplitude is $A^2 = y^2 + [\beta(s)y']^2$
- The change of the amplitude becomes

$$\delta(A^2) = \beta(s)\delta(y'^2) \Rightarrow A\delta A = -\beta^2(s)y'^2\frac{\delta E}{E}$$

- By averaging over all angles $\langle\beta^2(s)y'^2\rangle = \frac{A^2}{2\pi}\int_0^{2\pi}\sin^2\phi d\phi = \frac{A^2}{2}$ and $A\langle\delta A\rangle = -\frac{A^2}{2}\frac{\delta E}{E}$
- Summing up the energy losses for a full turn $\frac{\Delta A}{A} = -\frac{U_0}{2E}$
- Thus, in one turn the amplitudes are damped with a constant

$$\alpha_y = -\frac{\Delta A}{A\Delta t} = \frac{U_0}{2ET_0}$$

Εισαγωγή στη Φυσική των Επιταχυντών

Damping of vertical oscillations II

- The vertical betatron amplitude is thus exponentially decaying $$A(t) = A(0)e^{-\alpha_y t}$$
- Equivalently, the damping of the vertical emittance is given by $$\epsilon_y = \epsilon_y(0)e^{-2\alpha_y t}$$
- This means that the vertical emittance in the absence of dispersion or coupling will be reduced to zero
- Actually, due to the fact that the radiation emission is not parallel to the motion (but in an angular distribution with a width of ~1/γ), there is a "recoil" preventing vertical oscillations to be reduced to zero
- This gives a "quantum limit", beyond which the vertical emittance cannot be further reduced

Damping of horizontal oscillations

- The horizontal motion is described by

$$x = x_\beta + x_e = A\cos\phi + D\frac{\Delta E}{E} \quad \text{and} \quad x' = x'_\beta + x'_e = -\frac{A}{\beta(s)}\sin\phi + D'\frac{\Delta E}{E}$$

- Energy change $\mathcal{U}$due to photon emission results in a change of the dispersive part but not of the total coordinates so that

$$\delta x_\beta = -\delta x_e = D\frac{u}{E} \quad \text{and} \quad x\delta x'_\beta = -\delta x'_e = D'\frac{u}{E}$$

- The change of the Betatron amplitude $A^2 = x_\beta^2 + \left[\beta(s)x'_\beta\right]^2$ becomes $A\delta A = -(Dx_\beta + \beta^2(s)D'x'_\beta)\frac{u}{E}$
- The energy loss in an element *dl* is written

$$u = -\frac{P_s(x_\beta)}{c}dl = -\frac{1}{c}\left(P_s + 2\frac{P_s}{B}\frac{dB}{dx}x_\beta\right)\left(1 + \frac{x_\beta}{\rho}\right)ds$$

- Substituting to the change in amplitude and averaging over the angles (and some patience…)

$$\frac{\Delta A}{A} = -(1-\mathcal{D})\frac{U_0}{2E}$$ the damping coefficient $$\alpha_x = \frac{U_0}{2ET_0}(1-\mathcal{D})$$

Robinson theorem

- Grouping the damping constants and introducing the three damping times and damping partition numbers

$$\alpha_s = \frac{1}{\tau_s} = \frac{U_0}{2ET_0}(2+\mathcal{D}) = \frac{U_0}{2ET_0}\mathcal{J}_s$$

$$\alpha_y = \frac{1}{\tau_y} = \frac{U_0}{2ET_0} = \frac{U_0}{2ET_0}\mathcal{J}_y$$

$$\alpha_x = \frac{1}{\tau_x} = \frac{U_0}{2ET_0}(1-\mathcal{D}) = \frac{U_0}{2ET_0}\mathcal{J}_x$$

- The **Robinson** theorem (1958) states that the sum of the damping partition number is an invariant

$$\mathcal{J}_x + \mathcal{J}_y + \mathcal{J}_s = 4$$

- In storage ring with separated function magnets, $\mathcal{D} \ll 1$

$$\mathcal{J}_x = 1\ ,\quad \mathcal{J}_y = 1\ ,\quad \mathcal{J}_s = 2$$

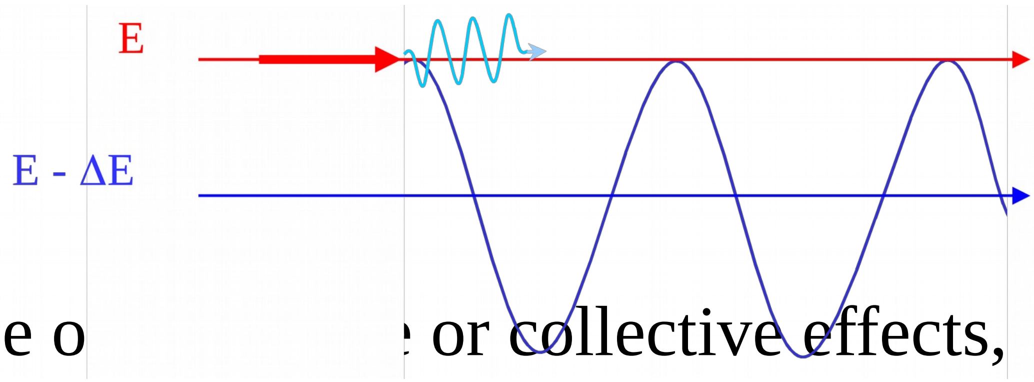
- The longitudinal damping occurs at twice the rate of the damping in the two transverse dimensions

Transverse emittances

- Radiation damping provides a direct mechanism to take "hot" injected beams and reduce the equilibrium parameters to a regime useful for high luminosity colliders and high brightness light sources.
- At the same time, the radiated power plays a dominant role in the design of the associated hardware and its protection
- If the only effect was radiation damping, the transverse emittances would be damped to zero.
- Photons are emitted in energy bursts in localized areas and horizontal betatron oscillations are excited as well (quantum fluctuations)
- Vertical emittance can become very small and only excited by coupling with the horizontal or residual vertical dispersion
- Electrons are influenced by this stochastic effect and eventually loose memory (unlike hadrons)

Equilibrium Beam Properties

- The emission of photons by the beam is a random process around the ring
- Photons are emitted within a cone around the direction of the beam particle with a characteristic angle *1/γ*
- This quantized process excites oscillations in each dimension



- In the absence o[…]e or collective effects, which also serve to **heat** the beam, the balance between quantum excitation and radiation damping results in the equilibrium beam properties that are characteristics of a given ring lattice

- For the very short timescales corresponding to photon emission, we can take the equations of motion we previously obtained for synchrotron motion and write:

$$\Delta E^2(t) + \frac{E^2\Omega^2}{\alpha_c^2}\tau^2(t) = A_E^2$$

where A_E is a constant of the motion.

- The change in A_E due to the emission of photons should be estimated
- The emission of an individual photon will not affect the time variable, however, it will cause an instantaneous change in the value of ΔE

Quantum Excitation - Longitudinal

- From the solution of the synchrotron equation of motion, the energy difference is

$$\delta(\Delta E) = A_0 \cos\Omega(t - t_0) - \frac{u}{E}\cos\Omega(t - t_1) = A_1 \cos\Omega(t - t_1)$$

where u is the energy radiated at time t_1. Thus, for $t=t_1$

$$A_1^2 = A_0^2 + \left(\frac{u}{E}\right)^2 - \frac{2A_0 u}{E}\cos\Omega(t_1 - t_0)$$

and $$\Delta A^2 = \langle A^2 - A_0^2 \rangle = \frac{u^2}{E^2}$$

- Considering the rate of photon emission $\mathcal{N}$, the average change in synchrotron amplitude due to photon emission is

$$\frac{d\langle A^2\rangle}{dt} = \mathcal{N}\left(\frac{u}{E_0}\right)^2$$

Quantum Excitation - Longitudinal

- By including, the radiation damping term, the net change in the synchrotron amplitude can be written as:

$$\frac{d\left\langle A^2 \right\rangle}{dt} = -2\alpha_E \left\langle A^2 \right\rangle + N \frac{u^2}{E^2}$$

- The equilibrium properties of a bunch are obtained when the rate of growth from quantum excitation and the rate of damping from radiation damping are equal
- For an ensemble of particles where the rms energy amplitude is represented by the rms energy spread, the equilibrium condition are written as

$$\sigma_\delta^2 = \left(\frac{\sigma_E}{E}\right)^2 = \frac{\left\langle A^2 \right\rangle}{2} = \frac{\left\langle N \left\langle u^2 \right\rangle \right\rangle_s}{4\alpha_E E^2}$$

Photon Emission

- The term $\left\langle \mathrm{N}\left\langle u^2\right\rangle\right\rangle_s$ is the ring-wide average of the photon emission rate, N, times the mean square energy loss associated with each emission

$$\mathrm{N} = \int_0^\infty n(u)\,du \quad \text{and} \quad \mathrm{N}\left\langle u^2\right\rangle = \int_0^\infty u^2 n(u)\,du$$

where $n(u)$ is the photon emission rate at energy u,

$$\left\langle N\left\langle u^2\right\rangle\right\rangle_s = \frac{1}{C}\oint N\left\langle u^2\right\rangle ds$$

with C is the ring circumference.

- The derivation of the photon spectrum emitted in a magnetic field is quite lengthy and we just quote the result

$$N\left\langle u^2\right\rangle = 2C_q\gamma^2\frac{E\,P_\gamma}{\rho} \quad \text{where} \quad C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} = 3.84\times10^{-13}\,m$$

Energy Spread and Bunch Length

- Integrating around the ring then yields the RMS beam energy spread

$$\sigma_\delta^2 = \left(\frac{\sigma_E}{E}\right)^2 = C_q \gamma^2 \frac{I_3}{J_s I_2} = C_q \gamma^2 \frac{I_3}{2I_2 + I_4} \quad \text{where} \quad I_3 = \oint \frac{ds}{|\rho|^3}$$

- Using this expression with the synchrotron equations of motion, the bunch length is related to the energy spread by

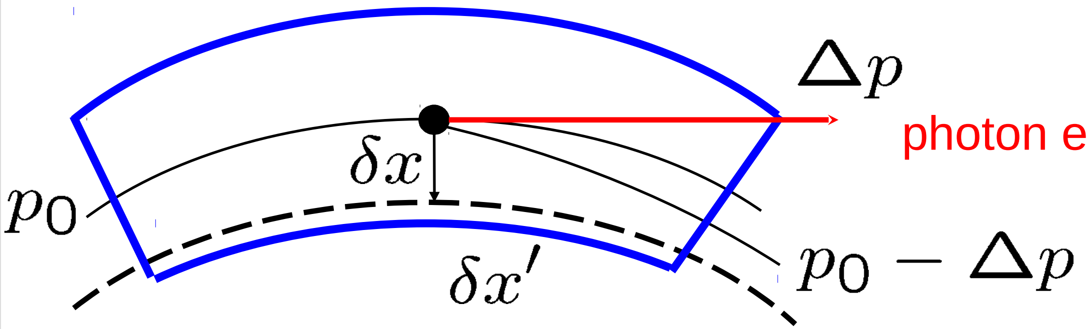
$$\sigma_z = \sigma_\delta \sqrt{\frac{\alpha_c C^2 \gamma m c^2}{2\pi h e V_0 |\cos\phi_s|}}$$

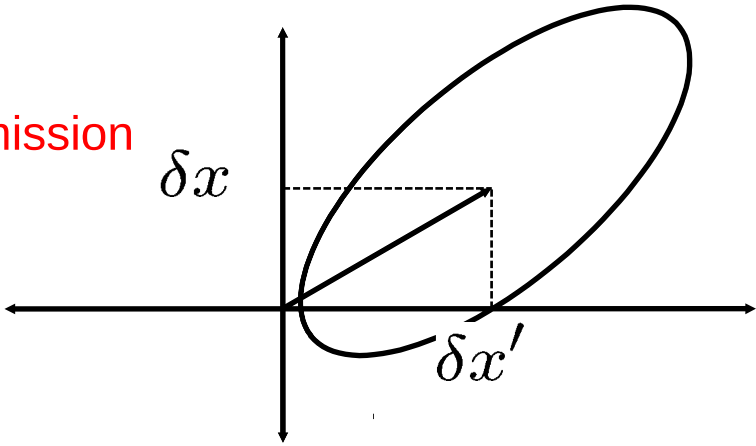
, with the harmonic number

$$h = \frac{f_{\mathrm{RF}} C}{c}$$

- The bunch length scales inversely with the square root of the RF voltage.

Quantum excitation - horizontal





- Assume electron along nominal momentum orbit with initially negligible emittance
- After photon emission with momentum Δp, electron's momentum becomes p_0-Δp and the trajectory becomes

$$\delta x = D\frac{\Delta p}{p} \quad \text{and} \quad \delta x' = D'\frac{\Delta p}{p}$$

Dispersion emittance

- Recall that the emittance of the betatron ellipse in phase space is

$$\varepsilon_x = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

- Taking into account the change of the position and angle due to the photon emission, the change of the emittance is

$$\delta\varepsilon_x = \left(\gamma D^2 + 2\alpha D D' + \beta D'^2\right)\left(\frac{\delta p}{p}\right)^2 = \mathcal{H}(s)\left(\frac{\delta p}{p}\right)^2$$

with the "**dispersion" emittance** (or curly H-function)

$$\boxed{\mathcal{H}(s) = \beta(s)D(s)'^2 + 2\alpha(s)D(s)D'(s) + \gamma(s)D(s)^2}$$

Horizontal equilibrium emittance

- Averaging over all photon energies and emission probabilities, the **equilibrium emittance** is derived as

$$\epsilon_x = \frac{C_q \gamma^2}{\mathcal{J}_x} \frac{\oint \frac{\mathcal{H}_x(s)}{|\rho_x|^3} ds}{\oint \frac{1}{\rho_x^2} ds}, \text{ with } \frac{C_q \gamma^2}{\mathcal{J}_x} \frac{\mathcal{I}_5}{\mathcal{I}_2}$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{h}{m_0 c} = 3.83 \times 10^{-13} \text{ m}$$

- For isomagnetic ring with separated function magnets the equilibrium emittance is written

$$\epsilon_x = 1.47 \times 10^{-6} \frac{E^2}{\rho} \frac{1}{l_{\mathrm{bend}}} \int_0^{l_{\mathrm{bend}}} \mathcal{H}_x(s) ds$$

- The integral depends on the optics functions on the bends
- It gets small for small horizontal beta and dispersion, but this necessitates strong quadrupoles
- Smaller bending angle and lower energy reduce equilibrium emittance

Quantum Excitation - Vertical

- In the vertical dimension, assuming an ideal ring with no vertical dispersion, the quantum excitation of the emittance is determined by the opening angle of the emitted photons. The resulting perturbation to the vertical motion can be described as:

$$\delta y = 0 \qquad \delta y' = \frac{\Delta p}{p}\theta_\gamma$$

and the effect to the emittance is

$$\delta \varepsilon_y = \left(\frac{\Delta p}{p}\theta_\gamma\right)^2 \beta_y$$

- Averaging over all photon energies and emission probabilities, the **quantum limit** of the **vertical emittance** is derived as

$$\varepsilon_y \approx \frac{C_q}{2 J_y I_2} \oint \frac{\beta_y}{\rho^3} ds$$

Vertical Emittance & Emittance Coupling

- For typical storage ring parameters, the vertical emittance due to quantum excitation is very small
- Assuming a typical β_y values of a few 10's of meters and bending radius of ~100 m, the quantum limit is $\varepsilon_y \sim 0.1$ pm.
- The observed sources of vertical emittance are:
  - **emittance coupling** whose source is ring errors which couple the vertical and horizontal betatron motion
  - **vertical dispersion** due to vertical misalignment of the quadrupoles and sextupoles and angular errors in the dipoles
- The vertical and horizontal emittances in the presence of a collection of such errors around a storage ring is commonly described as:

$$\varepsilon_y = \frac{\kappa}{1+\kappa}\varepsilon_0; \quad \varepsilon_x = \frac{1}{1+\kappa}\varepsilon_0 \quad \text{for} \quad 0 < \kappa < 1$$

ε_0 is the horizontal equilibrium (**natural**) emittance.

Radiation integrals

$$\mathcal{I}_1 = \oint \frac{D}{\rho} ds$$ Momentum compaction factor $$\boxed{\alpha_c = \frac{\mathcal{I}_1}{2\pi R}}$$

$$\mathcal{I}_2 = \oint \frac{1}{\rho^2} ds$$ Energy loss per turn $$\boxed{U_0 = \frac{C_\gamma}{2\pi} E^4 \mathcal{I}_2}$$

$$\mathcal{I}_3 = \oint \frac{1}{|\rho|^3} ds$$ Equilibrium energy spread $$\boxed{\sigma_\delta^2 = C_q \gamma^2 \frac{\mathcal{I}_3}{2\mathcal{I}_2 + \mathcal{I}_4}}$$

$$\mathcal{I}_4 = \oint \frac{D}{\rho^3}(1 + 2k\rho^2) ds$$ $$\boxed{\mathcal{J}_x = 1 - \frac{\mathcal{I}_4}{\mathcal{I}_2} \ , \ \ \mathcal{J}_s = 2 + \frac{\mathcal{I}_4}{\mathcal{I}_2} \ , \ \ \mathcal{D} = \frac{\mathcal{I}_4}{\mathcal{I}_2}}$$

Damping partition numbers

$$\mathcal{I}_5 = \oint \frac{\mathcal{H}}{|\rho^3|} ds$$ Equilibrium betatron emittance $$\boxed{\epsilon_x = C_q \gamma^2 \frac{\mathcal{I}_5}{\mathcal{I}_2 - \mathcal{I}_4}}$$