Εισαγωγή στη Φυσική των Επιταχυντών

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References

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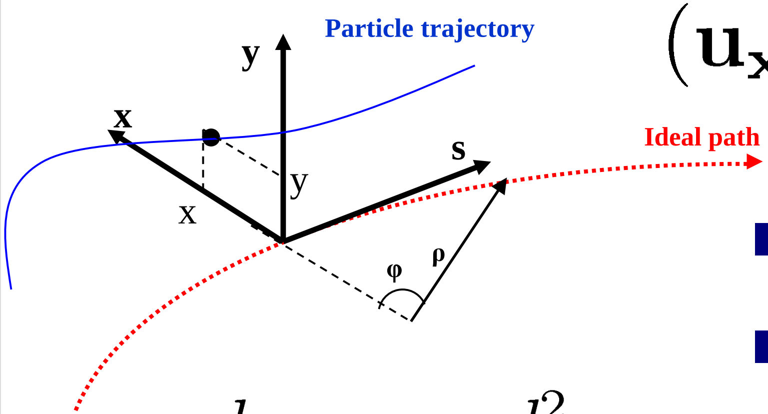
Outline - part I

- Coordinate system
- Equations of motion
- Hill's equations
  - Derivation
  - Harmonic oscillator
- Transport Matrices
  - Matrix formalism
  - Drift
  - Thin lens
  - Quadrupoles
  - Dipoles
    - Sector magnets
    - Rectangular magnets
  - Doublet
  - FODO

Coordinate system

- Cartesian coordinates not useful to describe motion in an accelerator

- Instead we use a system following an ideal path along the accelerator



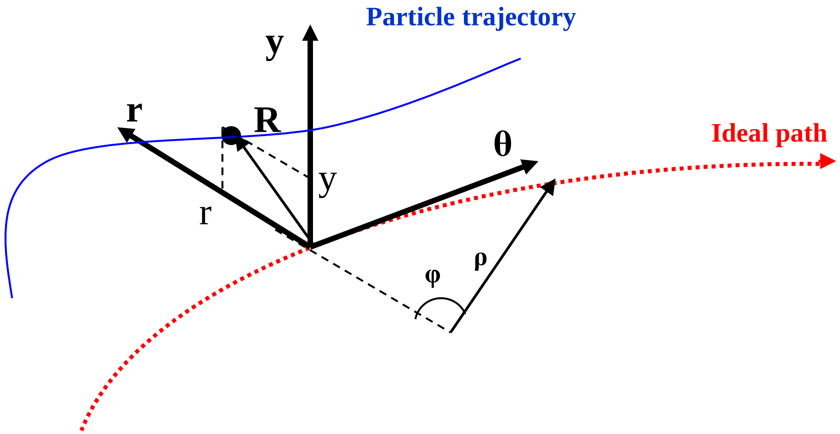
$$(\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_z}) \longrightarrow (\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_s})$$

- The curvature is $\mathbf{k} = -\dfrac{d^2\mathbf{s}}{ds^2}$
- From Lorentz equation

$$\frac{d\mathbf{p}}{dt} = m\gamma\frac{d^2\mathbf{s}}{dt^2} = m\gamma v_s^2\frac{d^2\mathbf{s}}{ds^2} = -m\gamma v_s^2\mathbf{k} = q|\mathbf{v}\times\mathbf{B}|$$

- The ideal path is defined $\mathbf{k} = -\dfrac{q}{p}|\dfrac{\mathbf{v}}{v_s}\times\mathbf{B}|$

Rotating coordinate system



- Consider a particle with charge *q* moving in the presence of transverse magnetic fields
- Choose cylindrical coordinate system *(r,φ,y)*, with *r* = *x+ρ* and *φ* = *s/ρ*
- The radius vector is

$$\mathbf{R} = \mathbf{R_0} + r\mathbf{u_r} + y\mathbf{u_y}$$

- For a small displacement **dφ**

$$d\mathbf{u_r} = d\phi\mathbf{u}_\phi \ , \quad d\mathbf{u}_\phi = -d\phi\mathbf{u_r} \ , \quad d\mathbf{u_y} = 0$$

- Than the velocity is $\dot{\mathbf{R}} = \dot{r}\mathbf{u_r} + r\dot{\phi}\mathbf{u}_\phi + \dot{y}\mathbf{u_y}$
- And the acceleration $\ddot{\mathbf{R}} = (\ddot{r} - r\dot{\phi}^2)\mathbf{u_r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\mathbf{u}_\phi + \ddot{y}\mathbf{u_y}$
- Recall that the momentum is $\dot{\mathbf{p}} = \frac{d}{dt}(\gamma m_0\dot{\mathbf{R}}) = \gamma m_0\ddot{\mathbf{R}}$

Equations of motion

- Setting the electric field to zero and the magnetic field

$$\mathbf{B} = (B_r, B_\phi, B_y) = (B_x, 0, B_y)$$

The Lorentz equations become

$$\dot{\mathbf{p}} = q\mathbf{v} \times \mathbf{B} = q\left[-r\dot{\phi}B_y\mathbf{u_r} + (\dot{y}B_x - \dot{r}B_y)\mathbf{u}_\phi + r\dot{\phi}B_x\mathbf{u_y}\right]$$

- Replacing the momentum with the adequate expression and splitting the equations for the *r* and *y* direction

$$\begin{aligned} \gamma m_0(\ddot{r} - r\dot{\phi}^2) &= -qr\dot{\phi}B_y \\ \gamma m_0\ddot{y} &= qr\dot{\phi}B_x \end{aligned}$$

- Replace $r\dot{\phi} = v_\phi$, $r = x + \rho$ and as $v_\phi >> v_r,\ v_y \rightarrow P/v_\phi \approx \gamma m_0$
- The equations of motion in the new coordinates are

$$\begin{aligned} \frac{P}{v_\phi}(\ddot{x} - \frac{v_\phi^2}{\rho + x}) &= -qv_\phi B_y \\ \frac{P}{v_\phi}\ddot{y} &= qv_\phi B_x \end{aligned}$$

General equations of motion

- Note that for $x<<\rho$ $\quad \frac{1}{\rho + x} = \frac{1}{\rho}(1 - \frac{x}{\rho})$
- It is convenient to consider the arc length **s** as the independent variable

$$ds = \rho d\phi = \rho\dot{\phi}dt = v_\phi \frac{\rho}{\rho + x}dt \approx v_\phi(1 - \frac{x}{\rho})dt$$

and $\quad \frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v_\phi(1 - \frac{x}{\rho})\frac{d}{ds}\ , \quad \frac{d^2}{dt^2} \approx v_\phi^2 \frac{d^2}{ds^2}$

- Denote $\quad \frac{dx}{ds} = x'\ , \ \frac{d^2x}{ds^2} = x''$
- The general equations of motion are

$$\boxed{\begin{aligned} x'' &= \frac{1}{\rho}(1 - \frac{x}{\rho}) - \frac{qB_y}{P} \\ y'' &= \frac{qB_x}{P} \end{aligned}}$$

- **Remark:** Note that without the approximations, the equations are nonlinear and coupled!
- The fields have to be defined

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Equations of motion – Linear fields

- Consider *s*-dependent fields from dipoles and normal quadrupoles $B_y = B_0(s) - g(s)x \ , \ \ B_x = -g(s)y$
- The total momentum can be written $P = P_0(1 + \frac{\Delta P}{P})$
- With magnetic rigidity $B_0\rho = \frac{P_0}{q}$ and normalized gradient

$k(s) = \frac{g(s)}{B_0\rho}$ the equations of motion are

$$\boxed{\begin{aligned} x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right) x &= \frac{1}{\rho(s)} \frac{\Delta P}{P} \\ y'' + k(s)\ y &= 0 \end{aligned}}$$

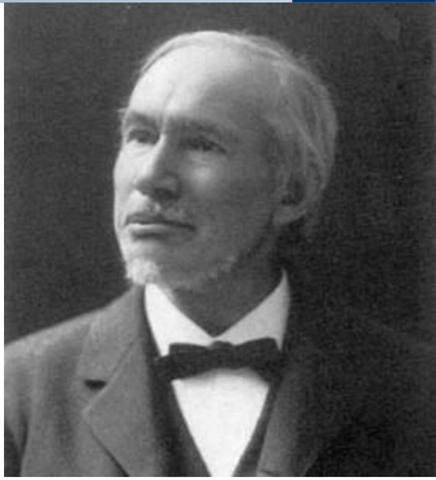
- Inhomogeneous equations with *s*-dependent coefficients
- Note that the term $1/\rho^2$ corresponds to the dipole **week focusing**
- The term $\Delta P/(P\rho)$ represents **off-momentum** particles

Hill's equations

- Solutions are combination of the ones from the homogeneous and inhomogeneous equations
- Consider particles with the design momentum. The equations of motion become

$$\boxed{\begin{aligned} x'' + K_x(s)\ x &= 0 \\ y'' + K_y(s)\ y &= 0 \end{aligned}}$$

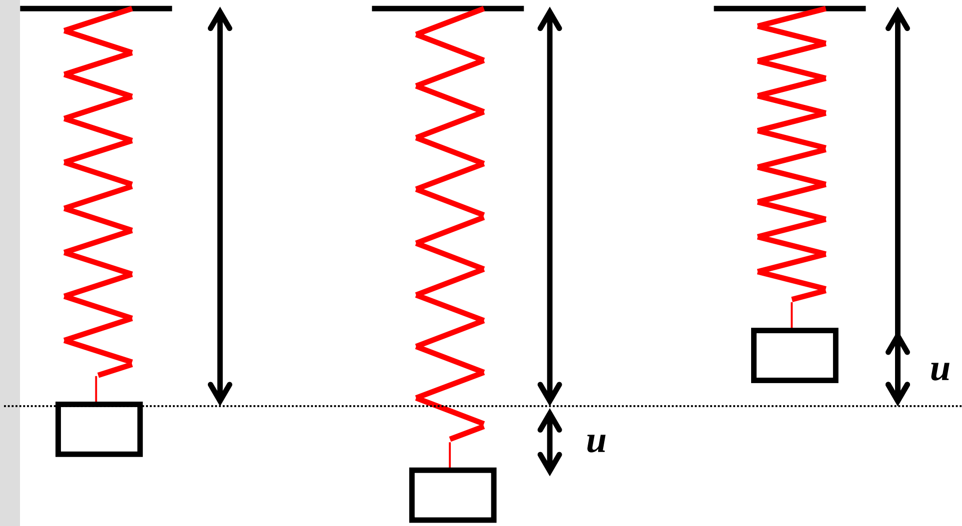
with $K_x(s) = -\left(k(s) - \dfrac{1}{\rho(s)^2}\right)\ ,\quad K_y(s) = k(s)$



George Hill

- **Hill's equations of linear transverse particle motion**
- Linear equations with s-dependent coefficients (harmonic oscillator with time dependent frequency)
- In a ring (or in transport line with symmetries), coefficients are periodic $K_x(s) = K_x(s+C)\ ,\quad K_y(s) = K_y(s+C)$
- Not straightforward to derive analytical solutions for whole accelerator

Harmonic oscillator – spring



- Consider $K(s) = k_0$ = constant

$$u'' + k_0 \; u = 0$$

- Equations of harmonic oscillator with solution

$$\begin{aligned} u(s) &= C(s)\; u(0) + S(s)\; u'(0) \\ u'(s) &= C'(s)u(0) + S'(s)u'(0) \end{aligned}$$

with

$$C(s) = \cos(\sqrt{k_0}\, s) \;, \quad S(s) = \frac{1}{\sqrt{k_0}} \sin(\sqrt{k_0}\, s) \qquad \text{for } k_0 > 0$$

$$C(s) = \cosh(\sqrt{|k_0|}\, s) \;, \quad S(s) = \frac{1}{\sqrt{|k_0|}} \sinh(\sqrt{|k_0|}\, s) \qquad \text{for } k_0 < 0$$

- Note that the solution can be written in **matrix** form

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

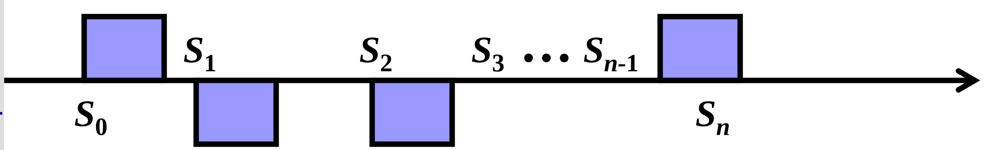
Matrix formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0)\begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

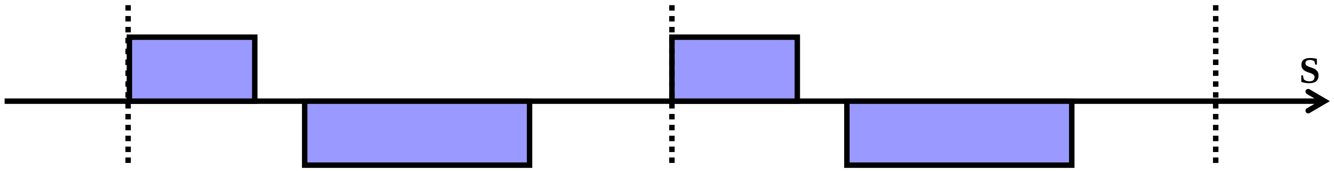
- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$ which is always true for conservative systems
- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$
- The accelerator can be build by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \underbrace{\mathcal{M}(s_n|s_{n-1})\ldots\underbrace{\mathcal{M}(s_3|s_2)\cdot\underbrace{\mathcal{M}(s_2|s_1)\cdot\underbrace{\mathcal{M}(s_1|s_0)}_{\text{from } s_0 \text{ to } s_1}}_{\text{from } s_0 \text{ to } s_2}}_{\text{from } s_0 \text{ to } s_3}}_{\text{from } s_0 \text{ to } s_n}$$



Symmetric lines

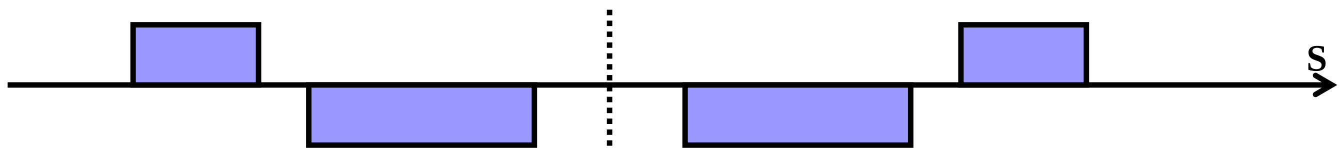
- System with normal symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M} \cdot \mathcal{M} = \mathcal{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

- System with mirror symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_r \cdot \mathcal{M} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{pmatrix}$$

4x4 Matrices

- Combine the matrices for each plane

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) \\ C'_x(s) & S'_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

to get a total 4x4 matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Uncoupled motion

Transfer matrix of a drift

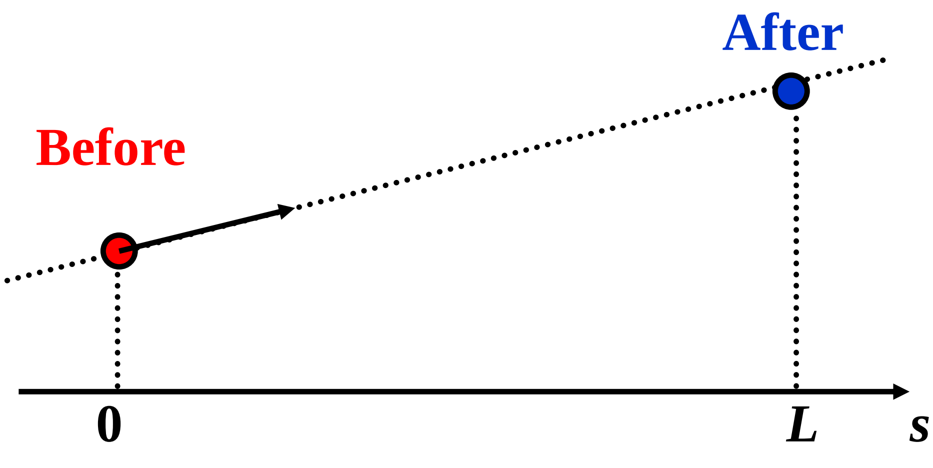
- Consider a drift (no magnetic elements) of length L=s-s_0

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

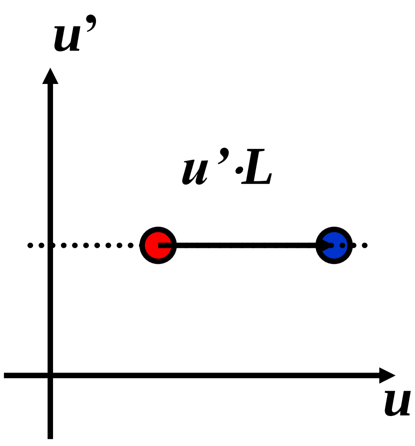
$$\boxed{\mathcal{M}_{\mathrm{drift}}(s|s_0) = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{aligned} u(s) &= u_0 + \overbrace{(s-s_0)}^{L} u'_0 = u_0 + L u'_0 \\ u'(s) &= u'_0 \end{aligned}$$

- Position changes if particle has a slope which remains unchanged.



Real Space



Phase Space

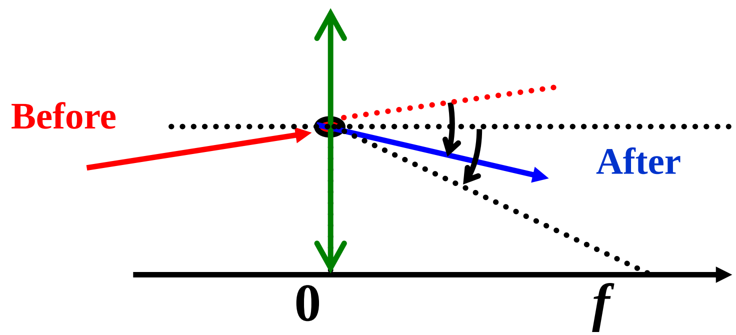
(De)focusing thin lens

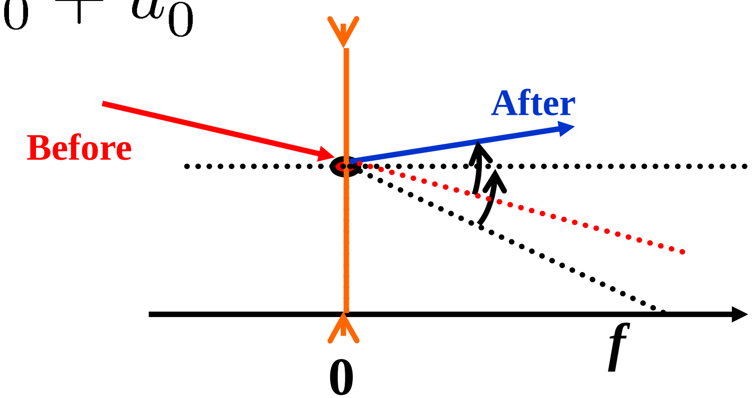
- Consider a lens with focal length $\pm f$

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

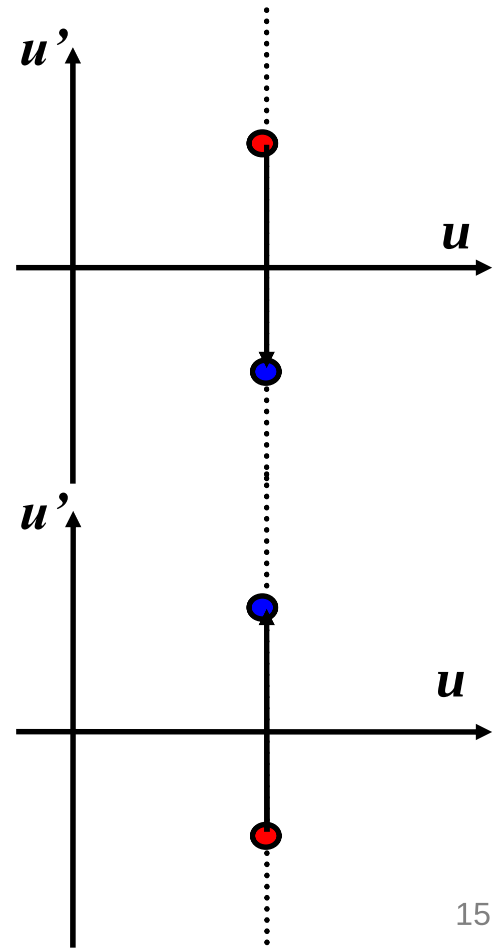
$$\boxed{\mathcal{M}_{\text{lens}}(s|s_0) = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}}$$

- Slope **diminishes** (focusing) or **increases** (defocusing) for positive position, which remains unchanged.





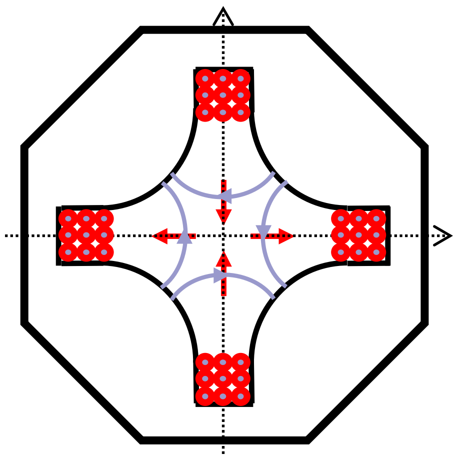
$$\begin{aligned} u(s) &= u_0 \\ u'(s) &= \mp\frac{1}{f}u_0 + u'_0 \end{aligned}$$



Quadrupole

- Consider a quadrupole magnet of length $L = s\text{-}s_0$. The field is

$$B_y = -g(s)x \ , \quad B_x = -g(s)y$$

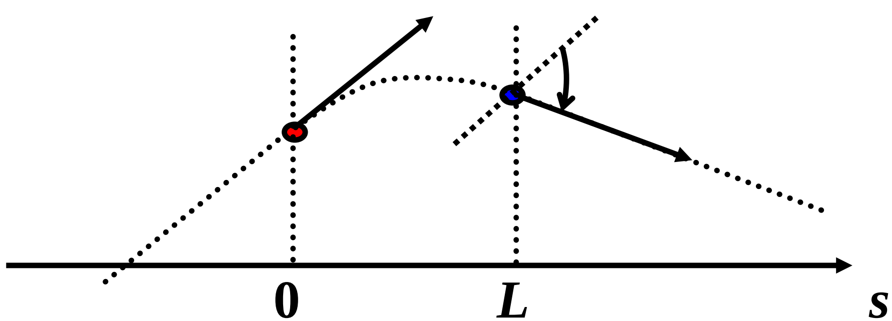


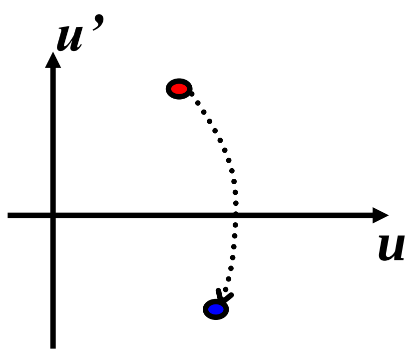
- with normalized quadrupole gradient (in $\mathbf{m^{-2}}$)

$$k(s) = \frac{g(s)}{B_0\rho}$$

The transport through a quadrupole is

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s-s_0)) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}(s-s_0)) \\ \sqrt{k}\sin(\sqrt{k}(s-s_0)) & \cos(\sqrt{k}(s-s_0)) \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$





(De)focusing Quadrupoles

- For a focusing quadrupole (k>0)

$$\boxed{\mathcal{M}_{\mathrm{QF}} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}L) \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}}$$

- For a defocusing quadrupole (k<0)

$$\boxed{\mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} \cosh(\sqrt{|k|}L) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}L) \\ \sqrt{|k|}\sinh(\sqrt{|k|}L) & \cosh(\sqrt{|k|}L) \end{pmatrix}}$$

- By setting $\sqrt{k}L \to 0$

$$\mathcal{M}_{\mathrm{QF,QD}} = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \mathcal{M}_{\mathrm{lens}}$$

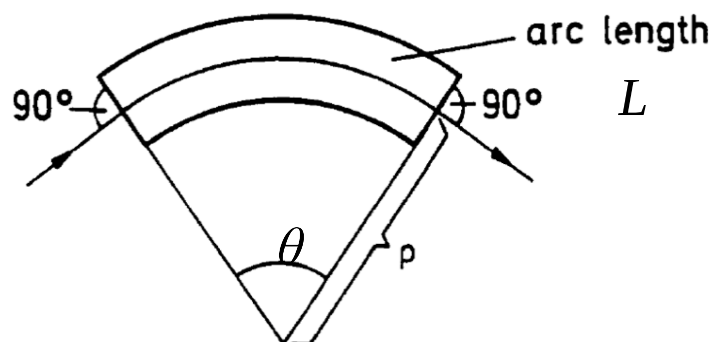
- **Note** that the **sign** of k or f is now absorbed inside the symbol
- In the other plane, focusing becomes defocusing and vice versa

Sector Dipole

- Consider a dipole of (arc) length L.
- By setting in the focusing quadrupole matrix $k = \dfrac{1}{\rho^2} > 0$ the transfer matrix for a sector dipole becomes

$$\boxed{\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}}$$

with a bending radius $\theta = \dfrac{L}{\rho}$

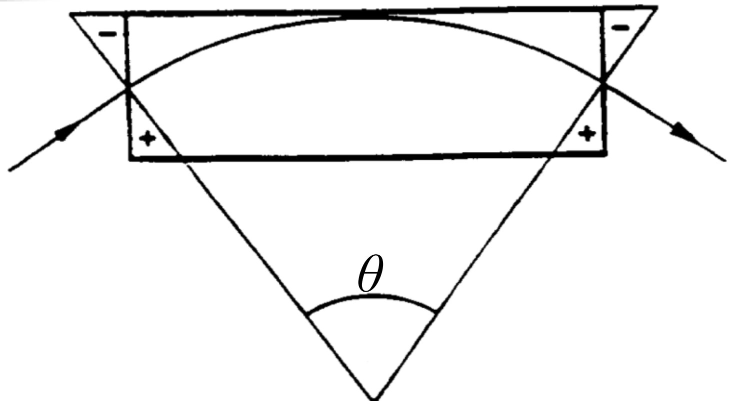


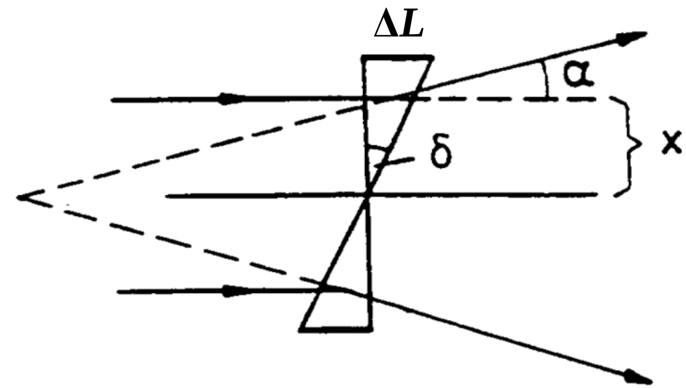
- In the non-deflecting plane $\dfrac{1}{\rho} \to 0$

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \mathcal{M}_{\text{drift}}$$

- This is a **hard-edge** model. In fact, there is some **edge focusing** in the vertical plane
- Matrix generalized by adding gradient (**synchrotron magnet**)

Rectangular Dipole





- Consider a rectangular dipole with bending angle θ. At each edge of length ΔL, the deflecting angle is changed by

$$\alpha = \frac{\Delta L}{\rho} = \frac{\theta \tan\delta}{\rho}$$

i.e., it acts as a thin defocusing lens with focal length $\frac{1}{f} = \frac{\tan\delta}{\rho}$

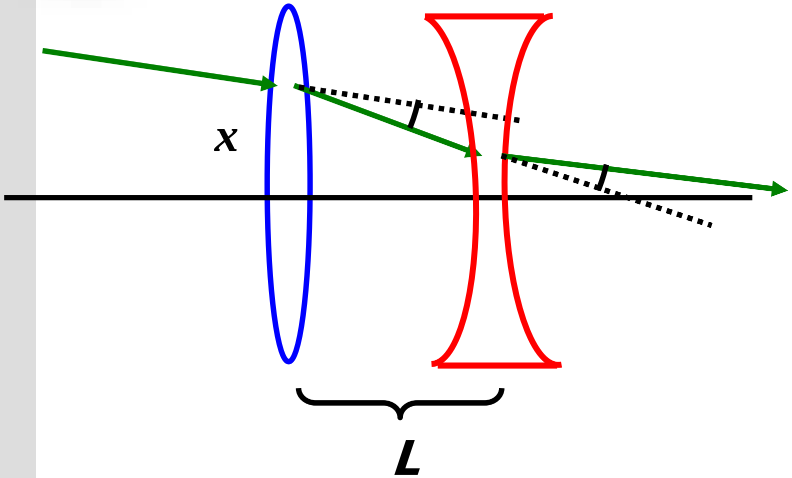
- The transfer matrix is $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sector}} \cdot \mathcal{M}_{\text{edge}}$ with

$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 \\ -\frac{\tan(\delta)}{\rho} & 1 \end{pmatrix}$$

- For **θ<<1**, **δ=θ/2**
- In deflecting plane (like **drift**), in non-deflecting plane (like **sector**)

$$\mathcal{M}_{x;\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \quad \mathcal{M}_{y;\text{rect}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

Quadrupole doublet



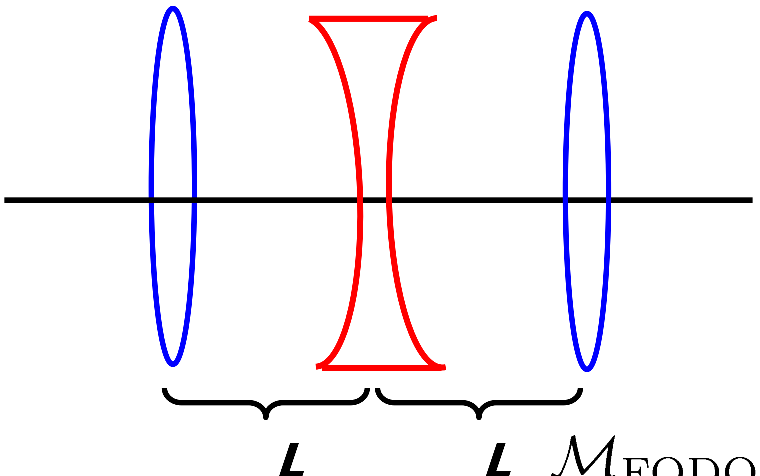
- Consider a quadrupole doublet, i.e. two quadrupoles with focal lengths f_1 and f_2 separated by a distance L.
- In thin lens approximation the transport matrix is

$$\mathcal{M}_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1-\frac{L}{f_2} \end{pmatrix}$$

with the **total focal length** $\dfrac{1}{f^\star} = \dfrac{1}{f_1} + \dfrac{1}{f_2} - \dfrac{L}{f_1\ f_2}$

- Setting $f_1 = -f_2 = f$ $\quad \dfrac{1}{f^\star} = \dfrac{L}{f^2}$
- **Alternating gradient focusing** seems overall focusing
- This is only valid in thin lens approximation

FODO Cell



- Consider defocusing quad "sandwiched" by two focusing quads with focal lengths ± *f*.
- Symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

with the transfer matrices

$$\mathcal{M}_{\mathrm{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} , \quad \mathcal{M}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} , \quad \mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

- The total transfer matrix is

$$\boxed{\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}}$$

Outline - part II

- General solutions of Hill's equations
  - Floquet theory
- Betatron functions
- Transfer matrices revisited
  - General and periodic cell
- General transport of betatron functions
  - Drift, beam waist
- Normalized coordinates
- Off-momentum particles
  - Effect from dipoles and quadrupoles
  - Dispersion equation
  - 3x3 transfer matrices

Solution of Betatron equations

- Betatron equations are linear

$$
\begin{aligned}
x'' + K_x(s)\, x &= 0 \\
y'' + K_y(s)\, y &= 0
\end{aligned}
$$

  with periodic coefficients

$$K_x(s) = K_x(s+C)\ , \quad K_y(s) = K_y(s+C)$$

- **Floquet theorem** states that the solutions are

$$u(s) = Aw(s)\cos(\psi(s) + \psi_0)$$

  where *w(s)*, *ψ(s)* are periodic with the same period

$$w(s) = w(s+C)\ , \quad \psi(s) = \psi(s+C)$$

- Note that solutions resemble the one of harmonic oscillator

$$u(s) = A\cos(\psi(s) + \psi_0)$$

- Substitute solution in Betatron equations

$$u'' + K(s)\, u = A\underbrace{(2w'\psi' + w\psi'')}_{0}\sin(\psi+\psi_0) + A\underbrace{(w'' - w\psi'^2 + Kw)}_{0}\cos(\psi+\psi_0) = 0$$

Betatron functions

- By multiplying with *w* the coefficient of sin

$$2w'w\psi' + w^2\psi'' = (w^2\psi')' = 0$$

- Integrate to get $\psi = \int \frac{ds}{w^2(s)}$

- Replace ψ' in the coefficient of cos and obtain

$$w^3(w'' + K_x w) = 1$$

- Define the **Betatron** or **Twiss** or **lattice functions** (Courant-Snyder parameters)

$$\boxed{\begin{aligned} \beta(s) &\equiv w^2(s) \\ \alpha(s) &\equiv -\frac{1}{2}\frac{d\beta(s)}{ds} \\ \gamma(s) &\equiv \frac{1+\alpha^2(s)}{\beta(s)} \end{aligned}}$$

Betatron motion

- The on-momentum linear betatron motion of a particle is described by

$$\boxed{u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \psi_0)}$$

with $\alpha,\ \beta,\ \gamma$ the twiss functions $\alpha(s) = -\frac{\beta(s)'}{2},\quad \gamma = \frac{1+\alpha(s)^2}{\beta(s)}$

ψ the **betatron phase** $\psi(s) = \int \frac{ds}{\beta(s)}$

and the **beta function** β is defined by the **envelope equation**

$$\boxed{2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4}$$

- By differentiation, we have that the **angle** is

$$\boxed{u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s) + \psi_0) + \alpha(s)\cos(\psi(s) + \psi_0)\right)}$$

Courant-Snyder invariant

- Eliminating the angles by the position and slope we define the **Courant-Snyder invariant**

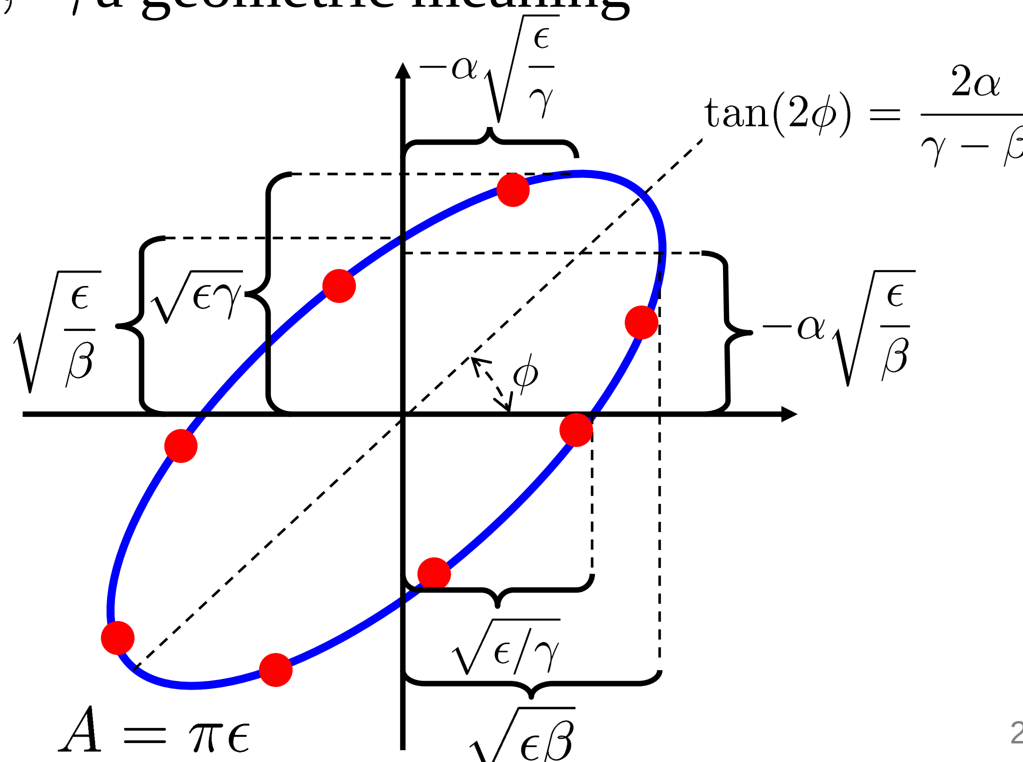
$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \epsilon$$

- This is an ellipse in phase space with area $\pi\varepsilon$
- The twiss functions $\alpha,\ \beta,\ \gamma$ a geometric meaning
- The beam envelope is

$$E(s) = \sqrt{\epsilon\beta(s)}$$

- The beam divergence

$$A(s) = \sqrt{\epsilon\gamma(s)}$$



General transfer matrix

- From equation for position and angle we have

$$\cos(\psi(s)+\psi_0)=\frac{u}{\sqrt{\epsilon\beta(s)}}\ ,\quad \sin(\psi(s)+\psi_0)=\sqrt{\frac{\beta(s)}{\epsilon}}u'+\frac{\alpha(s)}{\sqrt{\epsilon\beta(s)}}u$$

- Expand the trigonometric formulas and set $\psi(0)=0$ to get the transfer matrix from location 0 to s

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \mathcal{M}_{0\to s}\begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

with

$$\mathcal{M}_{0\to s}=\begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}(\cos\Delta\psi+\alpha_0\sin\Delta\psi) & \sqrt{\beta(s)\beta_0}\sin\Delta\psi \\ \frac{(a_0-a(s))\cos\Delta\psi-(1+\alpha_0\alpha(s))\sin\Delta\psi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\frac{\beta_0}{\beta(s)}}(\cos\Delta\psi-\alpha_0\sin\Delta\psi) \end{pmatrix}$$

and $\Delta\psi=\int_0^s\frac{ds}{\beta(s)}$ the **phase advance**

Periodic transfer matrix

- Consider a periodic cell of length *C*
- The optics functions are $\beta_0 = \beta(C) = \beta\ ,\quad \alpha_0 = \alpha(C) = \alpha$

  and the phase advance $$\mu = \int_0^C \frac{ds}{\beta(s)}$$

- The transfer matrix is

$$\mathcal{M}_C = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- The cell matrix can be also written as

$$\mathcal{M}_C = \mathcal{I}\cos\mu + \mathcal{J}\sin\mu$$

with $\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ne **Twiss matrix** $\boxed{\mathcal{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}$

Stability conditions

- From the periodic transport matrix $\text{Trace}(\mathcal{M}_C) = 2\cos\mu$ and the following stability criterion

$$\boxed{|\text{Trace}(\mathcal{M}_C)| < 2}$$

- In a ring, the **tune** is defined from the 1-turn phase advance

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

i.e. number betatron oscillations per turn

- From transfer matrix for a cell we get

$$\mathcal{M}_C = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\cos\mu = \frac{1}{2}(m_{11} + m_{22}),\ \beta = \frac{m_{12}}{\sin\mu},\ \alpha = \frac{m_{11} - m_{22}}{2\sin\mu},\ \gamma = -\frac{m_{21}}{\sin\mu}$$

Transport of Betatron functions

- For a general matrix between position 1 and 2

$$\mathcal{M}_{s_1\to s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$ and the inverse $$\mathcal{M}_{s_2\to s_1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

- Equating the invariant at the two locations

$$\epsilon = \gamma_{s_2} {u_{s_2}}^2 + 2\alpha_{s_2} u_{s_2} u'_{s_2} + \beta_{s_2} u'^2_{s_2} = \gamma_{s_1} {u_{s_1}}^2 + 2\alpha_{s_1} u_{s_1} u'_{s_1} + \beta_{s_1} u'^2_{s_1}$$

and eliminating the transverse positions and angles

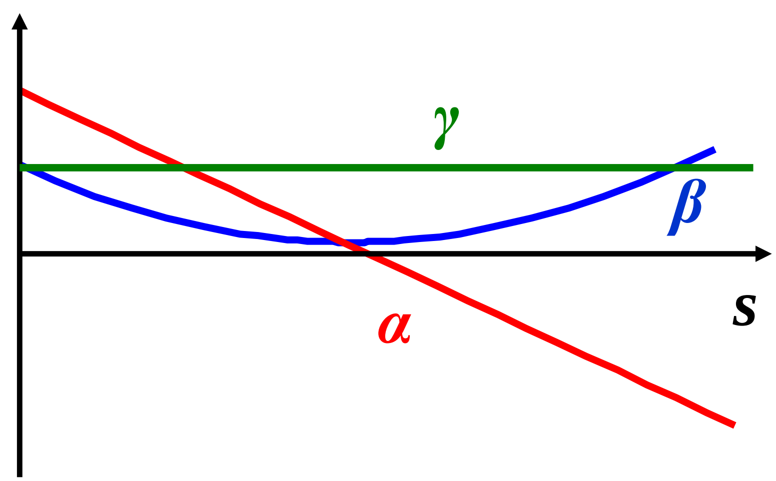
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{22}m_{12} \\ m_{21}^2 & 2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1}$$

Example I: Drift

- Consider a drift with length s
- The transfer matrix is $$\mathcal{M}_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$
- The betatron transport matrix is $$\begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix}$$

from which

$$\begin{aligned} \beta(s) &= \beta_0 - 2s\alpha_0 + s^2\gamma_0 \\ \alpha(s) &= \alpha_0 - s\gamma_0 \\ \gamma(s) &= \gamma_0 \end{aligned}$$



Simplified method for betatron transport

- Consider the beta matrix $\mathcal{B} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$ the matrix

$$\mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \text{ and its transpose } \mathcal{M}^T_{1\to 2} = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- It can be shown that

$$\boxed{\mathcal{B}_2 = \mathcal{M}_{1\to 2} \cdot \mathcal{B}_1 \cdot \mathcal{M}^T_{1\to 2}}$$

- Application in the case of the drift

$$\mathcal{B} = \mathcal{M}_{\text{drift}} \cdot \mathcal{B}_0 \cdot \mathcal{M}^T_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

and

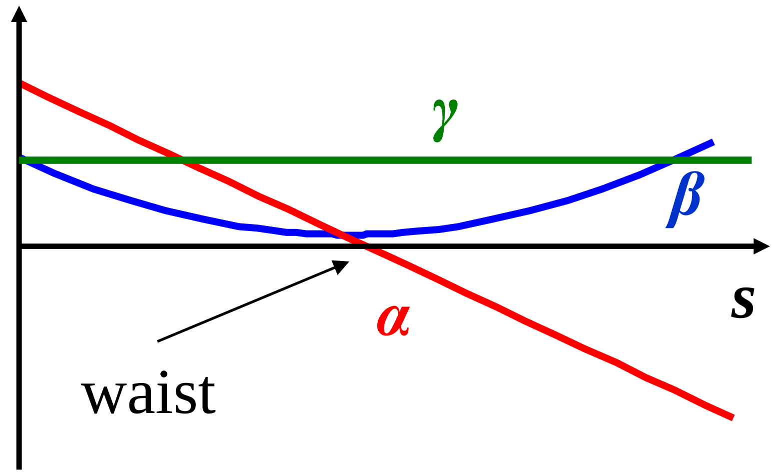
$$\mathcal{B} = \begin{pmatrix} \beta_0 - 2s\alpha_0 + s^2\gamma_0 & -\alpha_0 + s\gamma_0 \\ -\alpha_0 + s\gamma_0 & \gamma_0 \end{pmatrix}$$

Εισαγωγή στη Φυσική των Επιταχυντών

Example II: Beam waist

- For beam waist $\alpha=0$ and occurs at $s = \alpha_0/\gamma_0$
- Beta function grows quadratically and is minimum in waist

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$



- The beta at the waste for having beta minimum $\frac{d\beta(s)}{d\beta_0} = 0$

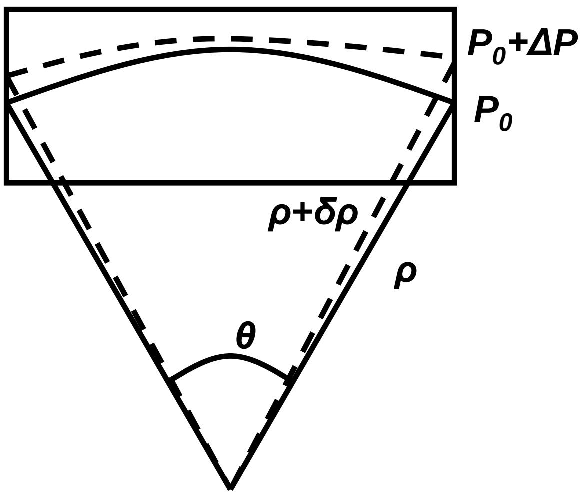
  in the middle of a drift with length L is $\beta_0 = \frac{L}{2}$

- The phase advance of a drift is $\mu = \int_0^{L/2} \frac{ds}{\beta(s)} = \arctan(\frac{L}{2\beta_0})$

  which is $\pi/2$ when $\beta_0 \to 0$. Thus, for a drift $\boxed{\mu = \leq \pi}$

Effect of dipole on off-momentum particles

- Up to now all particles had the same momentum P_0
- What happens for off-momentum particles, i.e. particles with momentum $P_0 + \Delta P$?
- Consider a dipole with field B and bending radius ρ
- Recall that the magnetic rigidity is $B\rho = \dfrac{P_0}{q}$ and for off-momentum particles



$$B(\rho + \Delta\rho) = \frac{P_0 + \Delta P}{q} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$

- Considering the effective length of the dipole unchanged

$$\theta\rho = l_{eff} = \text{const.} \Rightarrow \rho\Delta\theta + \theta\Delta\rho = 0 \Rightarrow \frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = -\frac{\Delta P}{P_0}$$

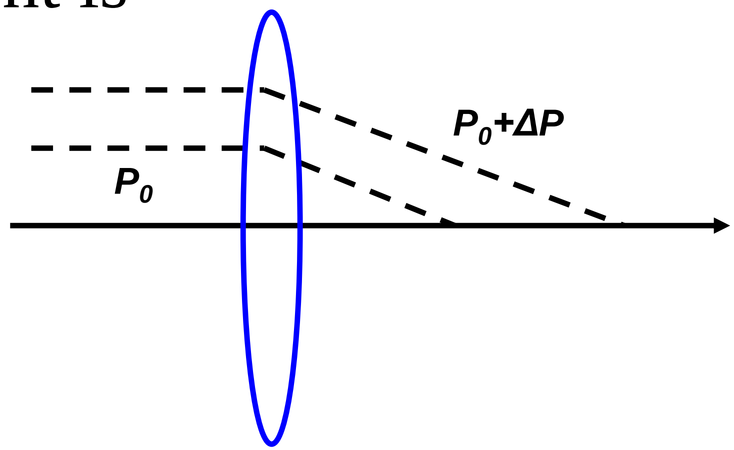
- Off-momentum particles get different deflection (different orbit)

$$\boxed{\Delta\theta = -\theta\frac{\Delta P}{P_0}}$$

Off-momentum particles and quadrupoles

- Consider a quadrupole with gradient ***G***
- Recall that the normalized gradient is

$$K = \frac{q\,G}{P_0}$$



and for off-momentum particles

$$\Delta K = \frac{dK}{dP}\Delta P = -\frac{qG}{P_0}\frac{\Delta P}{P_0}$$

- Off-momentum particle gets different focusing

$$\Delta K = -K\frac{\Delta P}{P_0}$$

- This is equivalent to the effect of **optical lenses** on **light of different wavelengths**

Dispersion equation

- Consider the equations of motion for off-momentum particles

$$x'' + K_x(s)x = \frac{1}{\rho(s)}\frac{\Delta P}{P}$$

- The solution is a sum of the **homogeneous** equation (on-momentum) and the **inhomogeneous** (off-momentum)

$$x(s) = x_H(s) + x_I(s)$$

- In that way, the equations of motion are split in two parts

$$\begin{aligned} x_H'' + K_x(s)x_H &= 0 \\ x_I'' + K_x(s)x_I &= \frac{1}{\rho(s)}\frac{\Delta P}{P} \end{aligned}$$

- The **dispersion function** can be defined as
- The dispersion equation is

$$D(s) = \frac{x_I(s)}{\Delta P/P}$$

$$\boxed{D''(s) + K_x(s)\ D(s) = \frac{1}{\rho(s)}}$$

Dispersion solution for a bend

- Simple solution by considering motion through a sector dipole with constant bending radius ρ
- The dispersion equation becomes $$D''(s) + \frac{1}{\rho^2} D(s) = \frac{1}{\rho}$$
- The solution of the homogeneous is harmonic with frequency $\frac{1}{\rho}$
- A particular solution for the inhomogeneous is $D_p = \text{constant}$ and we get by replacing $D_p = \rho$
- Setting $D(0) = D_0$ *and* $D'(0) = D_0'$, the solutions for dispersion are

$$\begin{aligned} D(s) &= D_0 \cos(\frac{s}{\rho}) + D_0' \rho \sin(\frac{s}{\rho}) + \rho(1 - \cos(\frac{s}{\rho})) \\ D'(s) &= -\frac{D_0}{\rho} \sin(\frac{s}{\rho}) + D_0' \cos(\frac{s}{\rho}) + \sin(\frac{s}{\rho}) \end{aligned}$$

General dispersion solution

- General solution possible with perturbation theory and use of Green functions
- For a general matrix $\mathcal{M} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S(s)' \end{pmatrix}$ the solution is

$$D(s) = S(s)\int_{s_0}^{s} \frac{C(\bar{s})}{\rho(\bar{s})} d\bar{s} + C(s)\int_{s_0}^{s} \frac{S(\bar{s})}{\rho(\bar{s})} d\bar{s}$$

- One can verify that this solution indeed satisfies the differential equation of the dispersion (and the sector bend)
- The general Betatron solutions can be obtained by 3X3 transfer matrices including dispersion

$$\mathcal{M}_{3\times 3} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

- Recalling that

$$x(s) = x_B(s) + D(s)\frac{\Delta P}{P}$$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \Delta p/p \end{pmatrix} = \mathcal{M}_{3\times 3} \begin{pmatrix} x(s_0) \\ x'(s_0) \\ \Delta p/p \end{pmatrix}$$ and $$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \mathcal{M}_{3\times 3} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

- For **drifts** and **quadrupoles** which do not create dispersion the 3x3 transfer matrices are just

$$\mathcal{M}_{\text{drift,quad}} = \begin{pmatrix} \mathcal{M}_{2\times 2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{pmatrix}$$

- For the deflecting plane of a **sector bend** we have seen that the matrix is

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and in the non-deflecting plane is just a drift.

3x3 transfer matrices - Synchrotron magnet

- Synchrotron magnets have focusing and bending included in their body.
- From the solution of the sector bend, by replacing *1/ρ* with

$$\sqrt{K} = \sqrt{\frac{1}{\rho^2} - k}$$

- For K>0 $$\mathcal{M}_{\mathrm{syF}} = \begin{pmatrix} \cos\psi & \frac{\sin\psi}{\sqrt{K}} & \frac{1-\cos\psi}{\rho K} \\ -\sqrt{K}\sin\psi & \cos\psi & \frac{\sin\psi}{\rho\sqrt{K}} \\ 0 & 0 & 1 \end{pmatrix}$$

- For K<0 $$\mathcal{M}_{\mathrm{syD}} = \begin{pmatrix} \cosh\psi & \frac{\sinh\psi}{\sqrt{|K|}} & -\frac{1-\cosh\psi}{\rho|K|} \\ \sqrt{|K|}\sinh\psi & \cosh\psi & \frac{\sinh\psi}{\rho\sqrt{|K|}} \\ 0 & 0 & 1 \end{pmatrix}$$

with $$\psi = \sqrt{|k + \frac{1}{\rho^2}|}\,l$$

3x3 transfer matrices - Rectangular magnet

- The end field of a rectangular magnet is simply the one of a quadrupole. The transfer matrix for the edges is

$$\mathcal{M}_{\rm edge} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan(\theta/2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

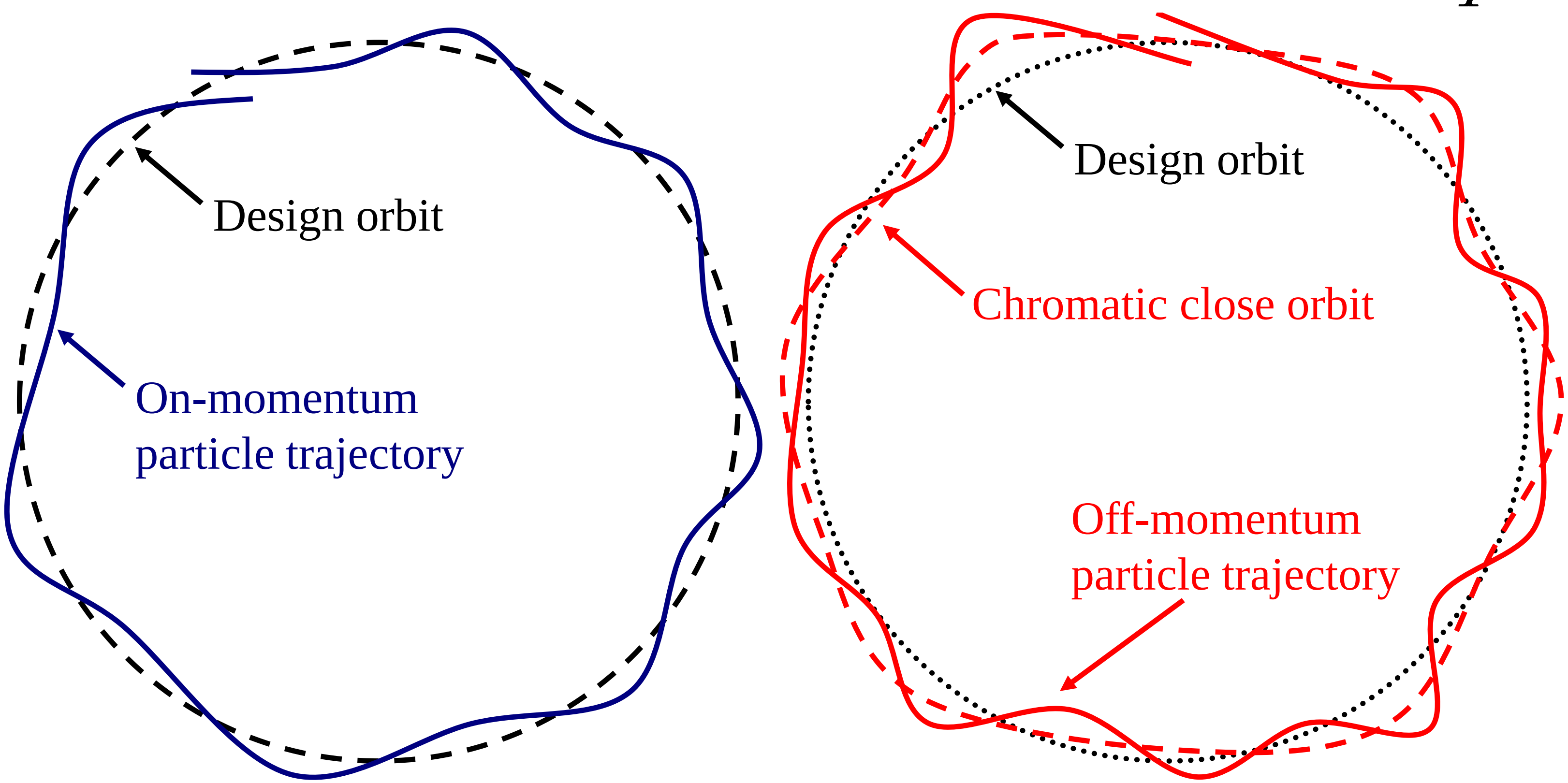
- The transfer matrix for the body of the magnet is like for the sector bend $\mathcal{M}_{\rm rect} = \mathcal{M}_{\rm edge} \cdot \mathcal{M}_{\rm sect} \cdot \mathcal{M}_{\rm edge}$
- The total transfer matrix is

$$\mathcal{M}_{\rm rect} = \begin{pmatrix} 1 & \rho\sin\theta & \rho(1-\cos\theta) \\ 0 & 1 & 2\tan(\theta/2) \\ 0 & 0 & 1 \end{pmatrix}$$

Chromatic closed orbit

- Off-momentum particles are not oscillating around design orbit, but around chromatic closed orbit
- Distance from the design orbit depends linearly with momentum spread and dispersion

$$x_D = D(s)\frac{\Delta P}{P}$$



Outline – part III

- Periodic lattices in circular accelerators
  - Periodic solutions for beta function and dispersion
  - Symmetric solution
- FODO cell
  - Betatron functions and phase advances
  - Optimum betatron functions
  - General FODO cell and stability
  - Solution for dispersion
  - Dispersion supressors
- General periodic solutions for the dispersion
- Tune and Working point
- Matching the optics

Periodic solutions

- Consider two points s_0 and s_1 for which the magnetic structure is repeated.
- The optical function follow periodicity conditions

$$\beta_0 = \beta(s_0) = \beta(s_1)\ , \quad \alpha_0 = \alpha(s_0) = \alpha(s_1)$$
$$D_0 = D(s_0) = D(s_1)\ , \ D'_0 = D'(s_0) = D'(s_1)$$

- The beta matrix at this point is $\mathcal{B}_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$
- Consider the transfer matrix from s_0 to s_1 $\quad \mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

$$\mathcal{B}_0 = \mathcal{M}_{0\to 1}\cdot\mathcal{B}_0\cdot\mathcal{M}_{0\to 1}^T \Rightarrow \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- The solution for the optics functions is

$$\beta_0 = \frac{2m_{12}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$
$$\alpha_0 = \frac{m_{11} - m_{22}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$

with the condition $\quad 2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2 > 0$

Periodic solutions for dispersion

- Consider the 3x3 matrix for propagating dispersion between s_0 and s_1

$$\begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

- Solve for the dispersion and its derivative to get

$$D_0' = \frac{m_{21}m_{13} + m_{23}(1 - m_{11})}{2 - m_{11} - m_{22}}$$

$$D_0 = \frac{m_{12}D_0' + m_{13}}{1 - m_{11}}$$

with the conditions $m_{11} + m_{22} \neq 2$ and $m_{11} \neq 1$

Symmetric solutions

- Consider two points s_0 and s_1 for which the lattice is mirror symmetric
- The optical function follow periodicity conditions

$$\alpha(s_0) = \alpha(s_1) = 0$$
$$D'(s_0) = D'(s_1) = 0$$

- The beta matrices at s_0 and s_1 are $\mathcal{B}_0 = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}$ $\mathcal{B}_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix}$
- Considering the transfer matrix between s_0 and s_1

$$\mathcal{B}_1 = \mathcal{M}_{0\to 1} \cdot \mathcal{B}_0 \cdot \mathcal{M}_{0\to 1}^T \Rightarrow \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- The solution for the optics functions is

$$\beta_0 = \sqrt{-\frac{m_{12}m_{22}}{m_{21}m_{11}}} \text{ and } \beta_1 = -\frac{1}{\beta_0}\frac{m_{12}}{m_{21}}$$

with the condition $\frac{m_{12}}{m_{21}} < 0 \text{ and } \frac{m_{22}}{m_{11}} > 0$

Symmetric solutions for dispersion

- Consider the 3x3 matrix for propagating dispersion between s_0 and s_1

$$\begin{pmatrix} D(s_1) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_0) \\ 0 \\ 1 \end{pmatrix}$$

- Solve for the dispersion in the two locations

$$\begin{aligned} D(s_0) &= -\frac{m_{23}}{m_{21}} \\ D(s_1) &= -\frac{m_{11}m_{23}}{m_{21}} + m_{13} \end{aligned}$$

- Imposing certain values for beta and dispersion, quadrupoles can be adjusted in order to get a solution

Periodic lattices' stability criterion revisited

- Consider a general periodic structure of length $2L$ which contains **N** cells. The transfer matrix can be written as

$$\mathcal{M}(s+N\cdot 2L|s) = \mathcal{M}(s+2L|s)^N$$

- The periodic structure can be expressed as

$$\mathcal{M} = \mathcal{I}\cos\mu + \mathcal{J}\sin\mu$$

with $$\mathcal{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} .$$

- Note that because $\det(\mathcal{M}) = 1 \rightarrow \beta\gamma - a^2 = 1$
- Note also that $\mathcal{J}^2 = -\mathcal{I}$
- By using **de Moivre's formula**

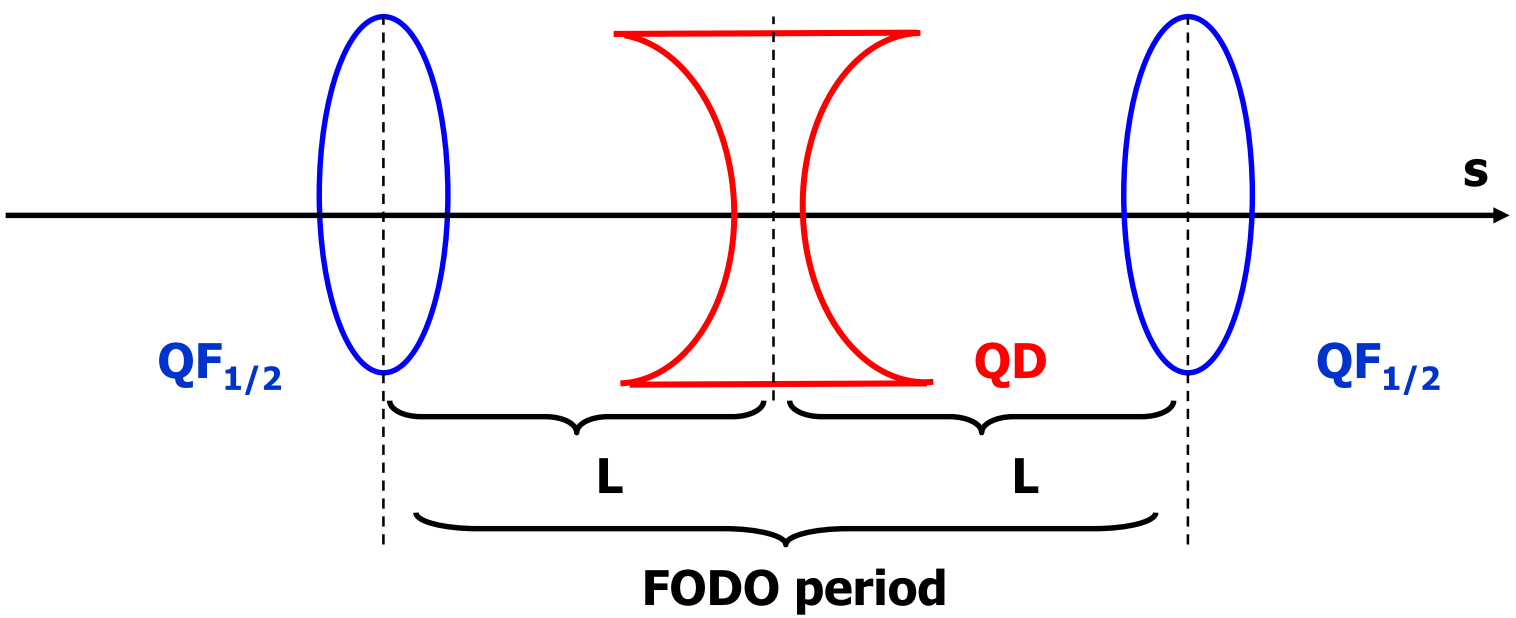
$$\mathcal{M}^N = (\mathcal{I}\cos\mu + \mathcal{J}\sin\mu)^N = \mathcal{I}\cos(N\mu) + \mathcal{J}\sin(N\mu)$$

- We have the following general stability criterion

$$|\mathrm{Trace}(\mathcal{M}^N)| = 2\cos(N\mu) < 2$$

FODO Cell

- FODO is the simplest basic structure
  - Half focusing quadrupole (F) + Drift (O) + Defocusing quadrupole (D) + Drift (O)
  - Dipoles can be added in drifts for bending
  - Periodic lattice with mirror symmetry in the center
  - Cell period from center to center of focusing quadrupole
  - The most common structure is accelerators



FODO transfer matrix

- Restrict study in thin lens approximation for simplicity
- FODO symmetric from any point to any point separated by 2L
- Useful to start and end at center of QF or QD, due to mirror symmetry
- The transfer matrix is

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

and we have

$$\boxed{\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L(1+\frac{L}{2f}) \\ \frac{L}{2f^2}(1-\frac{L}{2f}) & 1-\frac{L^2}{2f^2} \end{pmatrix}}$$

where we set $f_{\mathrm{QF}} = -f_{\mathrm{QD}} = f$ for a **symmetric FODO**

- Note that diagonal elements are equal due to mirror symmetry

$$\mathcal{M}_{\mathrm{tot}} = \mathcal{M}_r \cdot \mathcal{M} = \begin{pmatrix} ad+bc & 2bd \\ 2ac & ad+cb \end{pmatrix} \quad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

Betatron function for a FODO

- By using the formulas for the symmetric optics functions

$$\beta_0 = \frac{2m_{12}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}} \quad \text{and} \quad \alpha_0 = \frac{m_{11} - m_{22}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$

we get the beta on the center of the focusing quad

$$\beta^+ = L\frac{\kappa(\kappa+1)}{\sqrt{\kappa^2 - 1}} \quad \text{with} \quad \kappa = f/L > 1$$

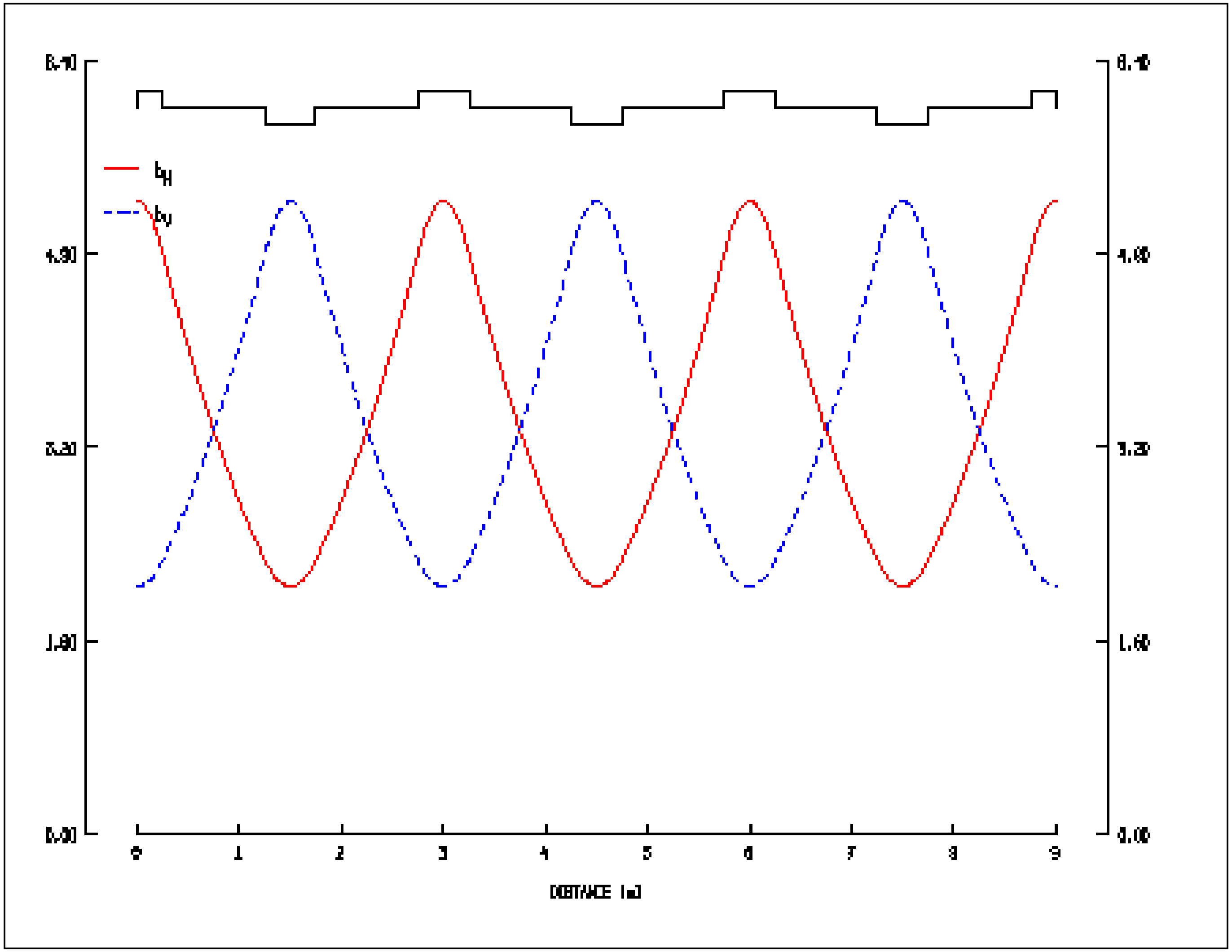
- Starting in the center of the defocusing quad (simply setting **f** to **-f**)

$$\beta^- = L\frac{\kappa(\kappa-1)}{\sqrt{\kappa^2 - 1}}$$

- Solutions for both horizontal and vertical plane
  - In the center of QF $\beta_x = \beta^+$ and $\beta_y = \beta^-$
  - In the center of QD $\beta_x = \beta^-$ and $\beta_y = \beta^+$
- Knowing the beta functions at one point, their evolution can be determined through the FODO cell

Example of Betatron functions evolution

- Betatron functions evolution in a FODO cell



Phase advance for a FODO

- For a symmetric cell, the transfer matrix can be written as

$$\mathcal{M}_{\text{sym}} = \begin{pmatrix} \cos\phi & \beta\sin\phi \\ -\frac{1}{\beta}\sin\phi & \cos\phi \end{pmatrix}$$

- So the phase advance is

$$\cos\phi = 1 - 2\frac{L^2}{f^2} = \frac{\kappa^2 - 2}{\kappa^2} \quad \text{or} \quad \sin\frac{\phi}{2} = \frac{1}{\kappa}$$

- This imposes the condition $\kappa > 1$ which means that the focal length should be smaller than the distance between quads
- For $\kappa \longrightarrow 1$, the beta function becomes infinite, so in between there should be a minimum

Optimum betatron functions in a FODO

- Start from the solution for beta in the focusing quad

$$\beta^+ = L\frac{\kappa(\kappa+1)}{\sqrt{\kappa^2-1}}$$

- Take the derivative to vanish $\frac{d\beta^+}{d\kappa} = 0 \rightarrow \kappa_0^2 - \kappa_0 - 1 = 0$
- The solution for the focusing strength is

$$\kappa_0 = \frac{1}{2} \pm \sqrt{\frac{1}{4}+1} \approx 1.6180$$

- So the optimum phase advance is $\phi_0 \approx 76.345^\circ$
- This solution however cannot minimize the betatron function in both planes
- It is good only for flat beams $\epsilon_x >> \epsilon_y$ or $\epsilon_y >> \epsilon_x$

Optimum betatron functions for round beams

- Consider a round beam $\epsilon_x \approx \epsilon_y$
- The maximum beam acceptance is obtained by minimizing quadratic sum of the envelopes
$$E_x^2 + E_y^2 = \epsilon_x \beta_x + \epsilon_y \beta_y \approx \epsilon(\beta_x + \beta_y)$$
- The minimum is determined by $\frac{d}{d\kappa}(\beta^+ + \beta^-) = 0$
- The minimum is reached for $\kappa_0 = \sqrt{2}$
and the optimum phase is $\phi_0 = 90^\circ$
- The betatron functions are $\beta^{\pm}_{\rm opt} = L(2 \pm \sqrt{2})$
- In order to fit an aperture of radius **R**
$$E_x^2 + E_y^2 = R^2 = \epsilon(\beta^+ + \beta^-) = \epsilon\ 4L$$
- The maximum emittance is $\epsilon_{\rm max} = \frac{R^2}{4L}$

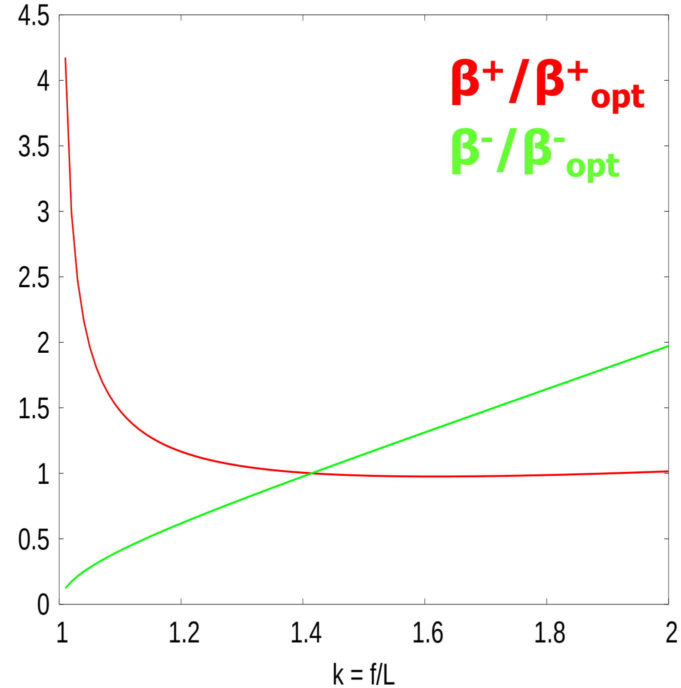
Εισαγωγή στη Φυσική των Επιταχυντών

Scaling of betatron functions in a FODO

- Scaling of the betatron functions with respect to the optimum values

$$\frac{\beta^+}{\beta^+_{\rm opt}} = \frac{\kappa(\kappa+1)}{(2+\sqrt{2})\sqrt{\kappa^2-1}} \quad \text{and} \quad \frac{\beta^-}{\beta^-_{\rm opt}} = \frac{\kappa(\kappa-1)}{(2-\sqrt{2})\sqrt{\kappa^2-1}}$$

- Scaling is independent of **L**
- It only depends on the ratio of the focal length and **L**
- The distance can be adjusted as a free parameter
- As the maximum beta functions are scaled linearly with **L**
- The maximum beam size in a FODO cell scales like with $\sqrt{L}$



Periodic lattices' stability criterion revisited

- Consider a general periodic structure of length 2L which contains **N** cells. The transfer matrix can be written as

$$\mathcal{M}(s+N\cdot 2L|s)=\mathcal{M}(s+2L|s)^N$$

- The periodic structure can be expressed as

$$\mathcal{M}=\mathcal{I}\cos\mu+\mathcal{J}\sin\mu$$

with $\mathcal{J}=\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}$.

- Note that because $det(\mathcal{M})=1\rightarrow\beta\gamma-a^2=1$
- Note also that $\mathcal{J}^2=-\mathcal{I}$
- By using **de Moivre's formula**

$$\mathcal{M}^N=(\mathcal{I}\cos\mu+\mathcal{J}\sin\mu)^N=\mathcal{I}\cos(N\mu)+\mathcal{J}\sin(N\mu)$$

- We have the following general stability criterion

$$|\mathrm{Trace}(\mathcal{M}^N)|=2\cos(N\phi)<2$$

General FODO cell

- So far considered transformation matrix for equal strength quadrupoles
- The general transformation matrix for a FODO cell

$$\mathcal{M}_{1/2} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1 - \frac{L}{f_2} \end{pmatrix}$$

with $\dfrac{1}{f^\star} = \dfrac{1}{f_1} + \dfrac{1}{f_2} - \dfrac{L}{f_1\ f_2}$

- Multiplication with the reverse matrix $\mathcal{M}^r_{1/2} = \begin{pmatrix} 1 - \frac{L}{f_2} & L \\ -\frac{1}{f^\star} & 1 - \frac{L}{f_1} \end{pmatrix}$ gives

$$\boxed{\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{2L}{f^\star} & 2L(1 - \frac{L}{f_2}) \\ -\frac{2}{f^\star}(1 - \frac{L}{f_1}) & 1 - \frac{2L}{f^\star} \end{pmatrix}}$$

Stability for a general FODO cell

- Setting $\frac{L}{f_1} = F$ and $-\frac{L}{f_2} = D$ we have hat the transfer matrix for a half cell is

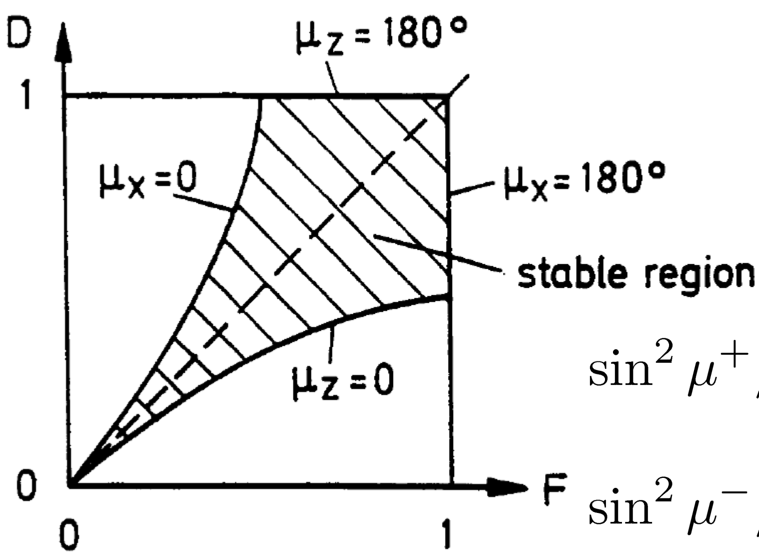
$$\mathcal{M}_{1/2\mathrm{FODO}} = \begin{pmatrix} 1-F & L \\ -\frac{1}{L}(F-D+FD) & 1+D \end{pmatrix}$$

- Equating this with the betatron transfer matrix we have

$$\mathcal{M}_{1/2FODO} = \begin{pmatrix} \sqrt{\frac{\beta^-}{\beta^+}}\cos\mu/2 & \sqrt{\beta^+\beta^-}\sin\mu/2 \\ -\frac{\sin\mu/2}{\sqrt{\beta^-\beta^+}} & \sqrt{\frac{\beta^+}{\beta^-}}\cos\mu/2 \end{pmatrix}$$

$$0 < F - D + FD = \sin^2\mu^+/2 < 1$$

$$0 < D - F + FD = \sin^2\mu^-/2 < 1$$



- The limits of the stable region give a **necktie**

$$\sin^2\mu^+/2 = 0 \rightarrow F = \frac{D}{1+D} \qquad \sin^2\mu^+/2 = 1 \rightarrow F = 1$$

$$\sin^2\mu^-/2 = 0 \rightarrow D = \frac{F}{1+F} \qquad \sin^2\mu^-/2 = 1 \rightarrow D = 1$$

3X3 FODO cell matrix

- Insert a sector dipole in between the quads and consider $\theta = L/\rho << 1$
- Now the transfer matrix is $\mathcal{M}_{\rm HFODO} = \mathcal{M}_{\rm HQF} \cdot \mathcal{M}_{\rm sector} \cdot \mathcal{M}_{\rm HQD}$ which gives

$$\mathcal{M}_{\rm HFODO} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{L^2}{2\rho} \\ 0 & 1 & \frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and after multiplication

$$\mathcal{M}_{\rm HFODO} = \begin{pmatrix} 1 - \frac{L}{f} & L & \frac{L^2}{(2\rho)} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} & \frac{L}{\rho}(1 + \frac{L}{2f}) \\ 0 & 0 & 1 \end{pmatrix}$$

Εισαγωγή στη Φυσική των Επιταχυντών

Dispersion in a FODO cell

- Consider mirror symmetry conditions, i.e. the dispersion derivative vanishes in the middle of quads

$$\begin{pmatrix} \eta^- \\ 0 \\ 1 \end{pmatrix} = \mathcal{M}_{\mathrm{HFODO}} \begin{pmatrix} \eta^+ \\ 0 \\ 1 \end{pmatrix}$$

- Solving for the dispersion in the entrance and exit

$$\eta^+ = \frac{L^2}{2\rho}\kappa(2\kappa+1) \quad \text{and} \quad \eta^- = \frac{L^2}{2\rho}\kappa(2\kappa-1) \quad \text{with} \quad \kappa = f/L$$

- We choose an optimum reference lattice where $\kappa_0 = \sqrt{2}$

$$\eta^+_{\mathrm{opt}} = \frac{L^2}{2\rho}(4+\sqrt{2}) \quad \text{and} \quad \eta^-_{\mathrm{opt}} = \frac{L^2}{2\rho}(4-\sqrt{2})$$

and the ratio $\displaystyle \frac{\eta^+}{\eta^+_{\mathrm{opt}}} = \frac{\kappa(2\kappa+1)}{(4+\sqrt{2})}$ and $\displaystyle \frac{\eta^-}{\eta^-_{\mathrm{opt}}} = \frac{\kappa(2\kappa-1)}{(4-\sqrt{2})}$

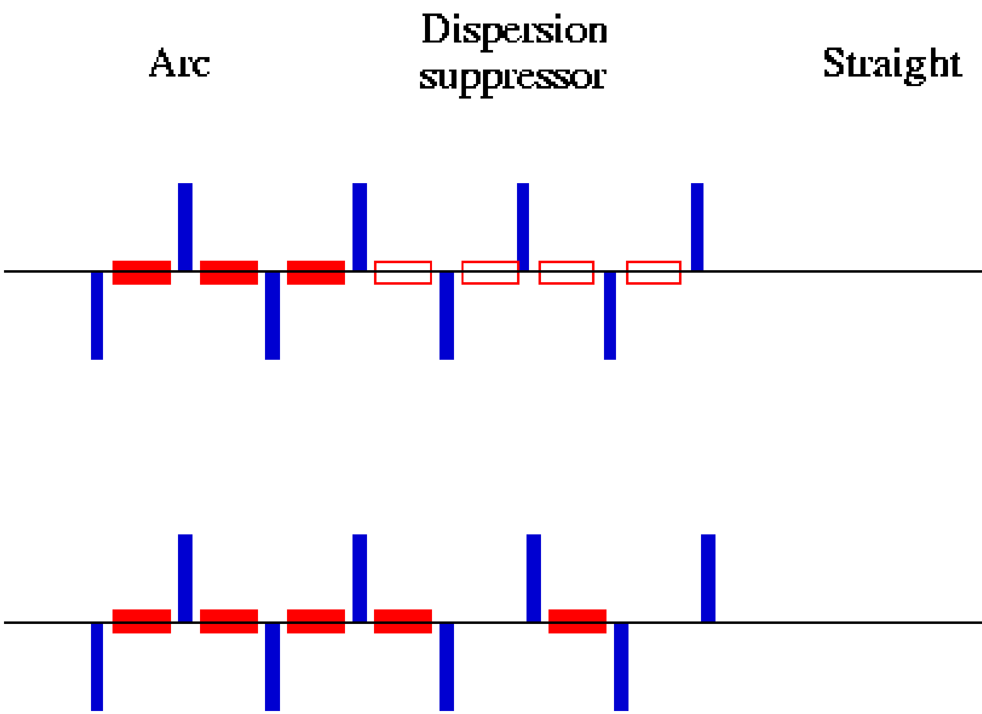
Dispersion suppressors

- Dispersion has to be eliminated in special areas like injection, extraction or interaction points (orbit independent to momentum spread)
- Use **dispersion suppressors**
- Two methods for suppressing dispersion
  - Eliminate two dipoles in a FODO cell (missing dipole)
  - Set last dipoles with different bending angles

$$\theta_1 = \theta\left(1 - \frac{1}{4\sin^2\mu_{\mathrm{HFODO}}}\right)$$

$$\theta_2 = \frac{\theta}{4\sin^2\mu_{\mathrm{HFODO}}}$$

  - For equal bending angle dipoles the FODO phase advance should be equal to **π/2**



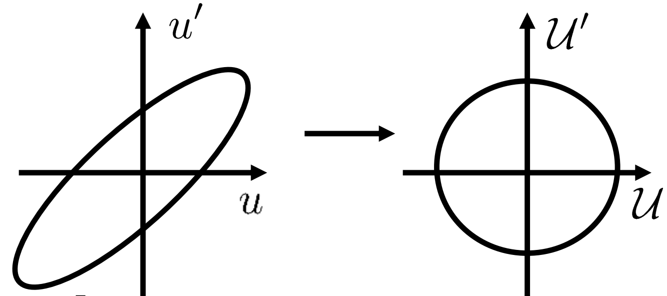
General solution for the dispersion

- Introduce **Floquet variables**

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\ ,\quad \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ ,\quad \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int\frac{ds}{\beta(s)}$$

- The Hill's equations are written $\displaystyle\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = 0$
- The solutions are the ones of an harmonic oscillator

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \sqrt{\epsilon}\begin{pmatrix}\cos(\nu\phi)\\ -\sin(\nu\phi)\end{pmatrix}$$



- For the dispersion solution $\mathcal{U} = \frac{D}{\sqrt{\beta}}\frac{\Delta P}{P}$, the inhomogeneous equation in Floquet variables is written

$$\frac{d^2 D}{d\phi^2} + \nu^2 D = -\frac{\nu^2\beta^{3/2}}{\rho(s)}$$

- This is a forced harmonic oscillator with solution

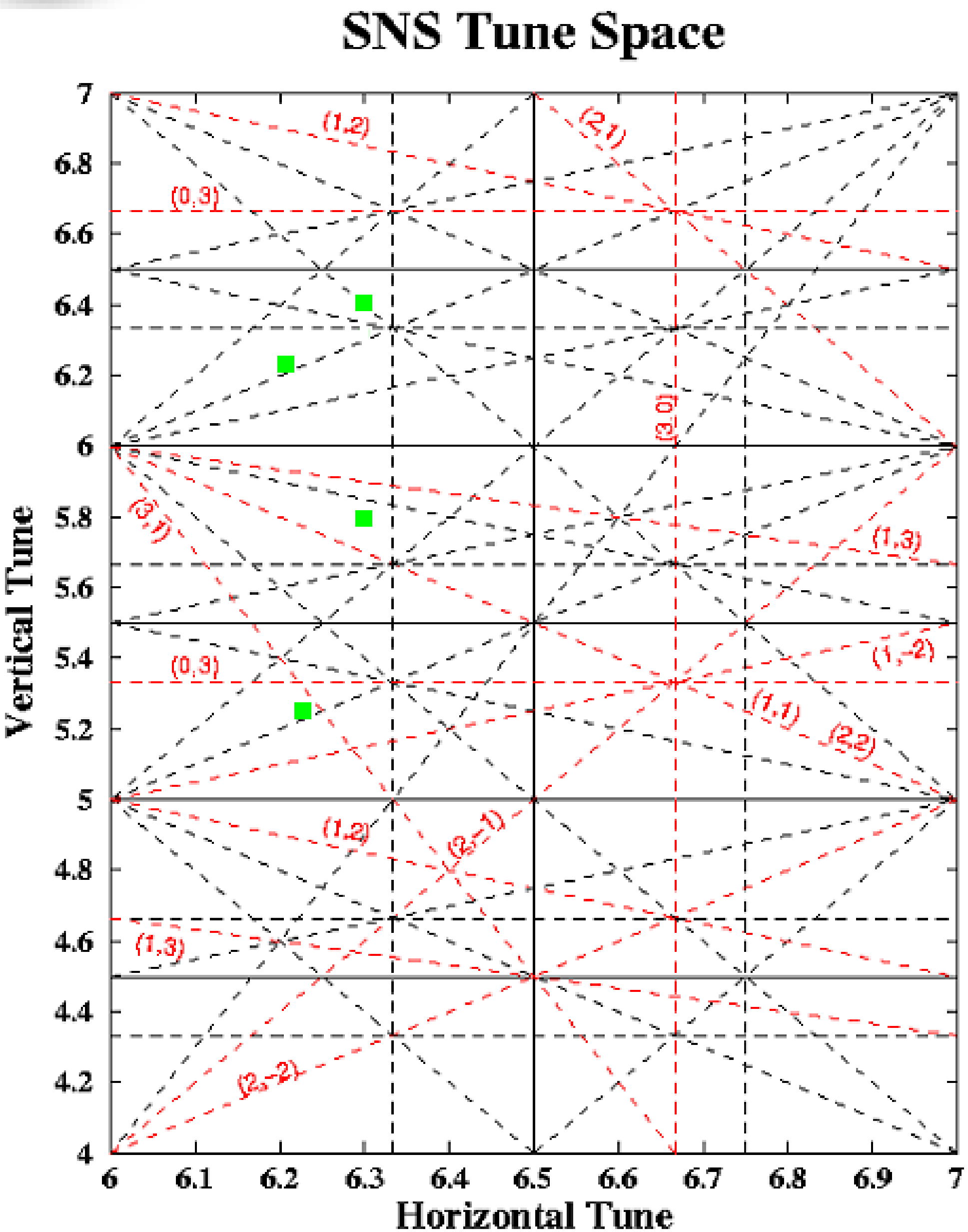
$$D(s) = \frac{\sqrt{\beta(s)}\nu}{2\sin(\pi\nu)}\oint\frac{\sqrt{\beta(\sigma)}}{\rho(\sigma)}\cos[\nu(\phi(s)-\phi(\sigma)+\pi)]d\sigma$$

- Note the **resonance conditions** for integer tunes!!!

Tune and working point

- In a ring, the **tune** is defined from the 1-turn phase advance
$$Q_{x,y} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)}$$
i.e. number betatron oscillations per turn
- Taking the average of the betatron tune around the ring we have in **smooth approximation**
$$2\pi Q = \frac{C}{\langle\beta\rangle} \rightarrow Q = \frac{R}{\langle\beta\rangle}$$
- Extremely useful formula for deriving scaling laws
- The position of the tunes in a diagram of horizontal versus vertical tune is called a **working point**
- The tunes are imposed by the choice of the quadrupole strengths
- One should try to avoid **resonance conditions**

Example: SNS Ring Tune Space



Tunability: 1 unit in horizontal, 3 units in vertical (2 units due to bump/chicane perturbation)

– **Structural resonances (up to 4th order)**

– **All other resonances (up to 3rd order)**

- **Working points considered**
  - **(6.30,5.80) - Old**
  - **(6.23,5.24)**
  - **(6.23,6.20) - Nominal**
  - **(6.40,6.30) - Alternative**

Matching the optics

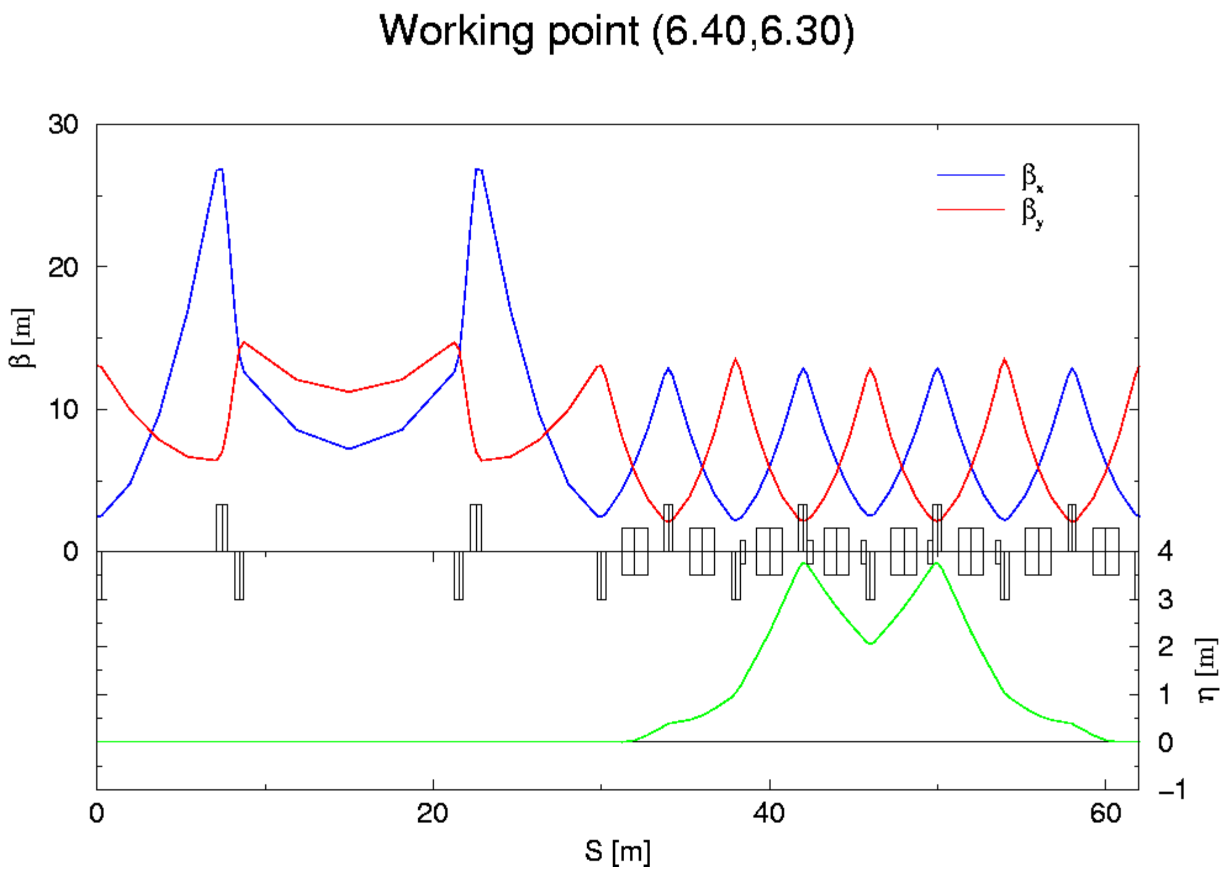
- Optical function at the **entrance** and **end** of accelerator may be fixed (pre-injector, or experiment upstream)
- Evolution of optical functions determined by magnets through transport matrices
- Requirements for aperture constrain optics functions all along the accelerator
- The procedure for choosing the quadrupole strengths in order to achieve all optics function constraints is called **matching of beam optics**
- Solution is given by numerical simulations with dedicated programs (MAD, TRANSPORT, SAD, BETA, BEAMOPTICS) through multi-variable minimization algorithms

magnet structure

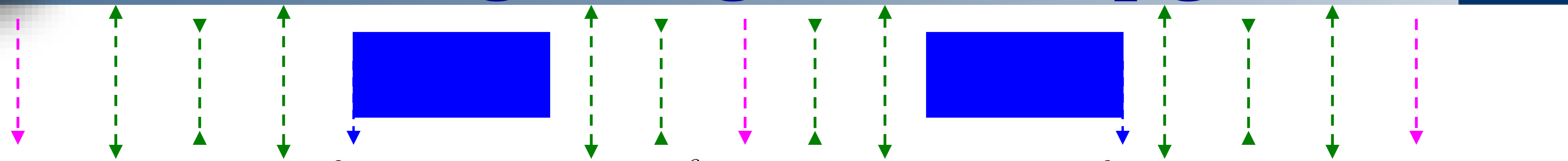
$$\begin{pmatrix}\beta(0)\\ \alpha(0)\\ \gamma(0)\end{pmatrix} \quad \mathbf{k_1} \quad \mathbf{k_2} \quad \mathbf{k_3} \quad \mathbf{k_4} \quad \mathbf{k_5} \quad \ldots \quad \mathbf{k_m} \quad \begin{pmatrix}\beta(\mathrm{end})\\ \alpha(\mathrm{end})\\ \gamma(\mathrm{end})\end{pmatrix}$$

Matching example – the SNS ring

- First find the strengths of the two arc quadrupole families to get an horizontal phase advance of **2π** and using the vertical phase advance as a parameter
- Then match the straight section with arc by using the two doublet quadrupole families and the matching quad at the end of the arc in order to get the correct tune without exceeding the maximum beta function constraints
- Retune arc quads to get correct tunes
- Always keep beta, dispersion within acceptance range and quadrupole strength below design values



ESRF storage ring lattice upgrade



β_{ID1} , $\alpha_{ID1} = 0$, $\eta_{ID1} = \sqrt{\beta_{ID1}\,\mathcal{H}}$, $\eta'_{ID1} = 0$

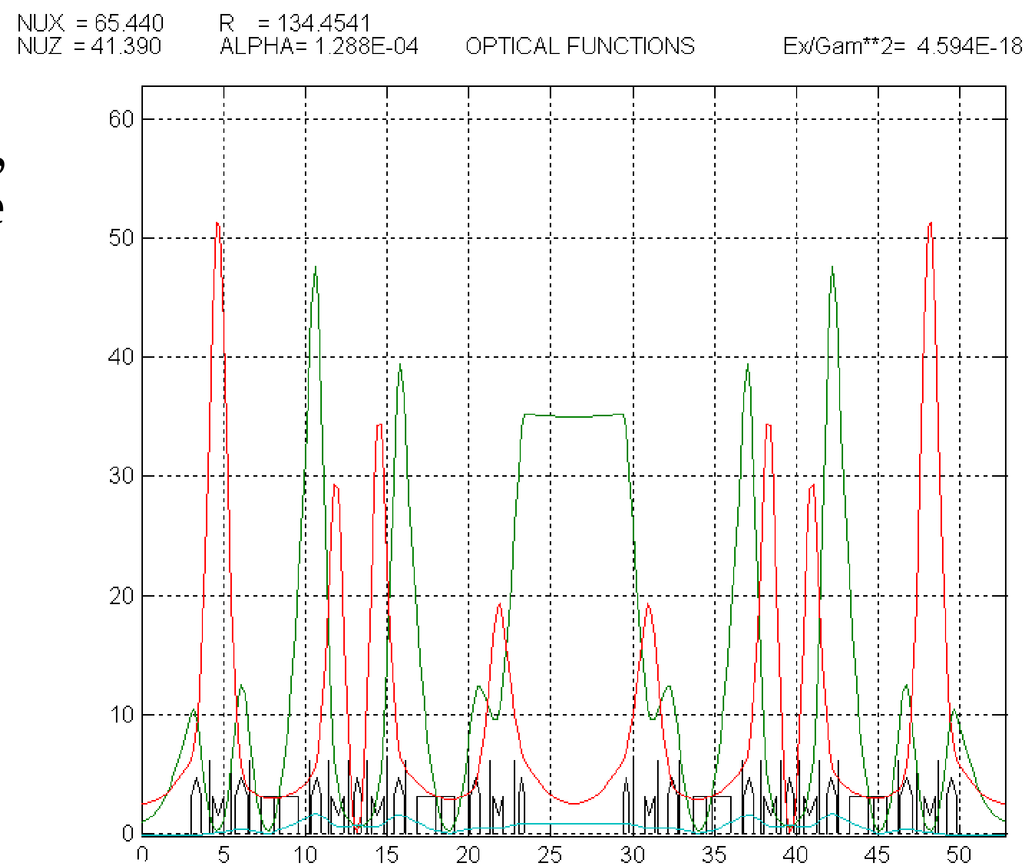
β_0 , α_0 , η_0 , η'_0

β_{SP} , $\alpha_{SP} = 0$, $\eta_{SP} = \sqrt{\beta_{SP}\,\mathcal{H}}$, $\eta'_{SP} = 0$

β_0 , α_0 , η_0 , η'_0

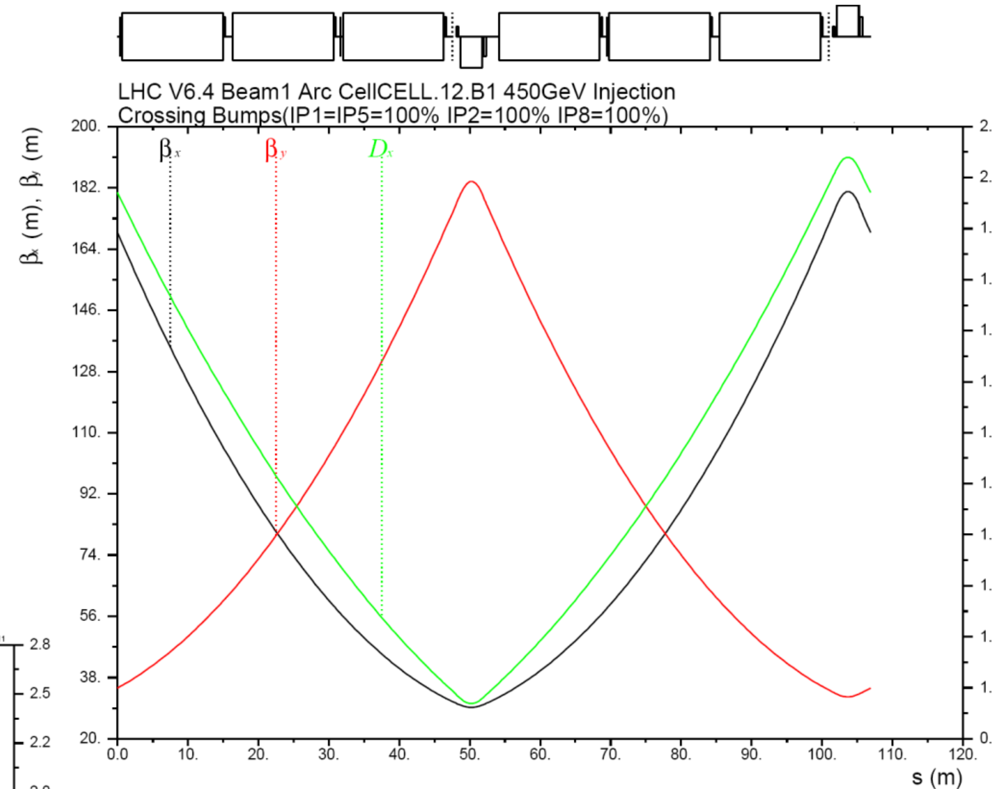
β_{ID2} , $\alpha_{ID2} = 0$, $\eta_{ID2} = \sqrt{\beta_{ID2}\,\mathcal{H}}$, $\eta'_{ID2} = 0$

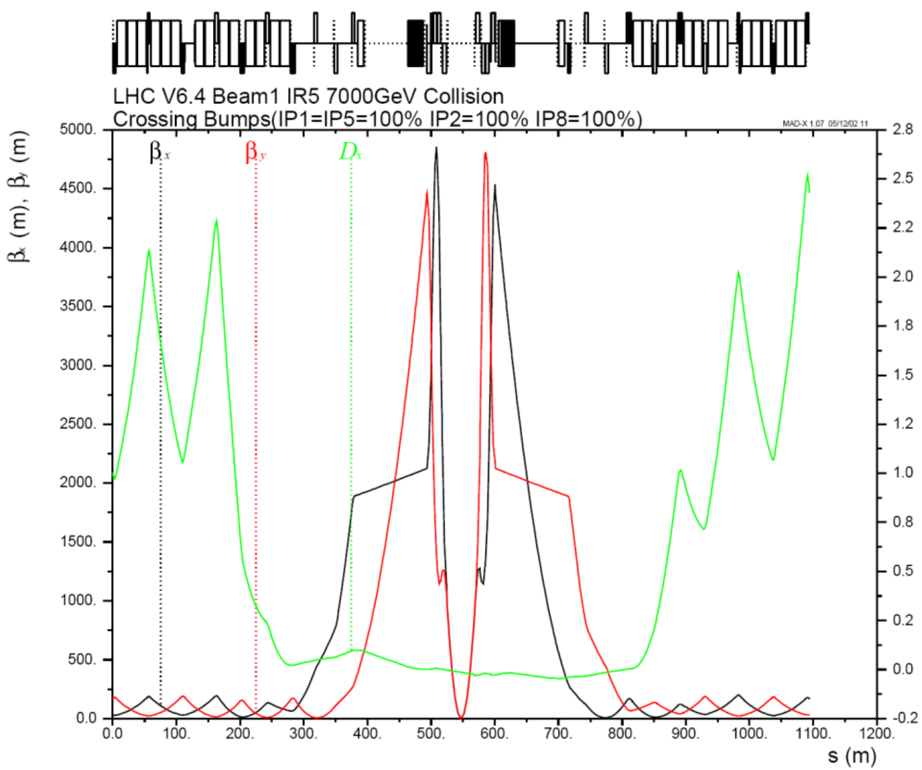
- Purpose to minimize emittance at the insertion device (increase brilliance) by imposing specific **β, α, D** and **D’** values at the entrance of the dipole
- Usually need to create achromat (dispersion equal to 0) in the straight section (**Double Bend Achromat – DBA, Triple Bend Achromat – TBA,…)**
- Try to minimize variation of beta function in the cell by tuning quadrupoles accordingly



LHC lattice examples

- FODO arc with 3+3 super-conducting bending magnets and 2 quadrupoles in between
- Beta functions between 30 and 180m





- Collision points creating beam waists with betas of **0.5m** using super-conducting quadrupoles in triplets
- Huge beta functions on triplets